# Optimal voting mechanisms with costly participation and abstention*

Hans Peter Grüner and Thomas Tröger

February 16, 2016

**Abstract**

How should a society choose between two social alternatives when valuations are private, monetary transfers are not feasible, and participation in the decision process is costly? We show that it is always socially optimal to use a linear voting rule: votes get alternative-dependent weights, and a default obtains if the weighted sum of votes stays below some threshold. A participation or approval quorum rule can be optimal only if one side of the electorate abstains. In the case of small participation costs, we characterize the equilibria of linear voting rules and solve for welfare-maximizing rules. Voluntary voting always dominates compulsory voting. If (and only if) the heterogeneity of preference intensities across the electorate is small, in the optimum essentially only one side of the electorate participates.

## 1 Introduction

Participating in collective decision procedures is typically individually costly. This is a non-trivial issue in modern societies where everybody is involved in numerous democratic decision procedures, not only at the community or state level,

*We thank Felix Bierbrauer, Luc Laeven, Andreas Kleiner, Igor Muraviev, Timofiy Mylovanov, Anne-Kathrin Roesler for helpful comments. Grüner, Tröger: University of Mannheim, Department of Economics, L7, 3-5 68131 Mannheim, Germany, gruener@uni-mannheim.de, troeger@uni-mannheim.de. We gratefully acknowledge financial support from the German Science Foundation (DFG) through SFB 884.”

but also as a firm employee or manager, a member of a professional organization, a trade union or a sports club, a parent of a high-school student, and so on.

From a welfare point of view, there is a trade-off between the desire to make the right choice and the desire to save on individuals' participation costs. In practice, a considerable variety of very different rules is used to deal with this trade-off. This includes voting rules with various majority thresholds, rules that only consider the number of affirmative votes, rules that grant veto rights, and majority rules that require an approval or a participation quorum. Moreover, there are mechanisms with voluntary and others with compulsory participation.

Börgers (2000, 2004) makes an important first step towards analyzing the role of different voting mechanisms. His setting is neutral across the two social alternatives. He shows that a voluntary majority rule yields a higher expected utility for each voter than a compulsory majority rule. Moreover, counting only the affirmative votes can be superior to a voluntary majority rule. But how should a decision-making mechanism *optimally* be designed when participation is costly? The present paper addresses this question. We provide a general analysis of the equilibria and welfare properties of voting mechanisms in private-value settings.

Our paper makes two main contributions. First, we show that optimality requires that the voting rule is outcome-equivalent to a linear rule. Linearity means that the votes for either alternative are weighted and a default obtains if the weighted sum of votes stays below some threshold (cf. Figures 1 and 2). Our second main contribution is to fully solve for the optimal linear voting rules in the case where voting costs are small. This part of our paper includes a complete analysis of equilibria of linear rules and their welfare properties.

We derive our results from a Bayesian model that is symmetric across the individuals that constitute the electorate. We consider arbitrary rules with simultaneous moves for choosing between two alternatives, S(tatus quo) and R(eform), such that the probability of choosing either alternative is a function of the votes cast in favor of each alternative, and there is no possibility to discriminate between voters. Formally, a rule stipulates the probability of choosing R as a function of the number of votes for S and of the number of votes for R (the remaining voters abstain). Each individual ("voter") is privately informed about her preferences over the social alternatives, and about her preference intensity relative to the "cost" of voting, which — based on a properly chosen Bernoulli representation of preferences — can be captured by a one-dimensional parameter ("type"). Each rule yields a Bayesian game in which each voter chooses among three actions: vote for S, vote for R, abstain. Abstaining saves the voting cost. We seek a rule together with a symmetric Bayesian Nash equilibrium that maximizes each individual's ex-ante expected utility; more generally, we consider Pareto efficiency with respect to the different types' interim expected utilities.

Our model subsumes a great variety of voting rules, including majority rules with arbitrary thresholds, rules that involve default alternatives or lotteries, as well as more exotic rules. For example, the probability of choosing an alternative may be strictly increasing in its vote share relative to the other alternative, such as in a proportional power sharing system. There may be an approval quorum or a participation quorum or other non-monotonic features. The features just mentioned may be combined. We do *not* consider rules with sequential moves (cf. Bognar et al. (2015)).[1] While we do not consider voting in a committee, our analysis is easily extended to incorporate this possibility.[2]

Let us explain the first main contribution. A rule is linear if there exists a weight for the S-votes, a weight for the R-votes, and a (possibly negative) default bias, such that the social alternative R is implemented if the weighted sum of R-votes minus the weighted sum of S-votes exceeds the default bias, and $S$ is implemented if the difference of the weighted sums falls below the default bias. We show that optimal rules are outcome-equivalent to linear rules. In fact, an optimum can always be found in the finite set of *upper* linear rules, which are defined by the additional property that $R$ is also implemented if the difference *equals* the default bias. The linear rules with default bias 0 are the (voluntary, qualified) majority rules. The linear rules in which one of the alternatives has weight 0 are such that this alternative serves as the default which prevails unless the other alternative gets enough votes ("one-sided default rules"). Equal weights for the two types of votes together with a positive default bias means that the R-voters need to beat the S-voters by a certain margin.

Quorum rules are the most prominent way used in practice to take account of voting costs.[3] According to a participation quorum rule, $R$ is implemented if

[1] Any sequential voting procedure can be symmetrized across voters with a random device that determines the order of moves. Our feeling is that administration of (and controlling the flow of information in) a sequential voting procedure is substantially more complex than with simultaneous moves. Transforming a sequential procedure into its normal form means that, from an individual's interim point of view, the payment of the participation cost can be uncertain and be dependent on others' actions; in our view, this contrasts the spirit of autonomous participation decisions.

[2] It is easy to see that forming a committee is not welfare-improving if the participation cost is small, but can be welfare-improving in general (Börgers (2000)).

[3] From the outset of democracy, quorum rules have been applied in collective decision making. According to Das (2014), the "city state of Athens, which arguably is the first jurisdiction to have instituted democratic institutions in around 508 BC, had its citizen assembly (ekklesia) at the heart

and only if (i) the number of $R$-votes is sufficiently large relative to number of $S$-votes and (ii) the number of votes cast in total exceeds a certain quota. In the event a single additional $S$-voter is needed to reach the quota, casting her vote may change the implemented alternative from $S$ to $R$. Thus, in contrast to linear rules, participation quorum rules are non-monotonic.[4] According to an approval quorum rule, $R$ is implemented if and only if condition (i) above holds and if (ii') the number of $R$-votes exceeds a certain quota. These rules are non-linear except in pathological cases. Due to non-linearity, a quorum rule can be optimal only together with an equilibrium in which one side of the electorate abstains completely; then only condition (ii') is relevant and the quorum rule is outcome-equivalent to a one-sided default rule. In any environment in which optimality requires that both sides of the electorate participate with positive probability, any non-linear quorum rule is suboptimal.

Our second main contribution refers to settings with small participation costs. We develop a perturbation method that facilitates a characterization of all equilibria of arbitrary voting rules. We apply this method to the upper linear rules and derive the first-order welfare effect of introducing a participation cost for each of the (up to three) relevant equilibria of such rules. Restricting attention to mechanism-equilibrium pairs that are optimal in the 0-participation-cost limit and comparing the corresponding first-order welfare effects allows us identify optimal and suboptimal linear voting rules. In this context, we obtain a number of fundamental conclusions concerning optimal voting with small participation costs.

First, the optimal voluntary voting rule is superior to the best compulsory voting rule. This is a far-reaching generalization of Börger's (2004) initial result.

Second, it can be optimal to have essentially only one side of the electorate participate in the voting or to have essentially everybody participate. Which of the two cases occurs depends on the degree of heterogeneity of preference intensities across the electorate. If the heterogeneity (appropriately measured) falls below a certain threshold, essentially only one side optimally participates. If it exceeds the threshold, essentially everybody optimally participates and the optimal rule is almost one-sided. Almost-one-sidedness means that, to upset the default alternative, a certain number of votes against it are required, and this number is lowered by 1 if there are exactly 0 votes in favor of the default alternative.

---

of its political life. The assembly used to gather at regular intervals (about 36 times a year) in a large stadium. The quorum requirement was 6000 citizens which was frequently met." See also Corte-Real and Pereira (2004) and Herrera and Matozzi (2008).

[4]Monotonicity means that increasing the number of votes in favor of an alternative cannot lower the probability that this alternative is chosen.

Third, if the heterogeneity is sufficiently large, then it is welfare-improving to artificially increase the voting cost.[5] Fourth, if one side of the electorate is to abstain, it should be the supporters of the majority alternative, provided the electorate's bias in favor of this alternative is sufficiently strong.

In some applications, the social alternatives are fundamentally symmetric, like two new candidates running for an office. Then only rules that are neutral in the sense of treating both alternatives identically may be considered legitimate. In the final section of the paper, we consider optimality across neutral rules and equilibria in neutral environments. An environment is neutral across the alternatives if switching the alternatives' labels leaves the voters' preferences unchanged. We show that in the neutral setting the standard voluntary majority rule is optimal among all neutral voluntary rules; this result does *not* require a small participation cost. The result can be combined with Börgers' (2004) comparison of voluntary and compulsory majority voting to conclude that optimal neutral voluntary voting dominates optimal compulsory voting in the neutral setting.

On a technical level, we make use of two alternative strategies of expressing equilibria: (i) in terms of the two pivot probabilities and (ii) in terms of the two probabilities that a voter participates and votes $S$ or $R$. The pivot probability of an R-vote is the expected change in probability that the alternative R is selected if a single voter switches from abstaining to voting for R. The pivot probability of an S-vote is defined similarly. The welfare can be conveniently expressed in terms of (i) the probability of implementing reform if a given voter abstains and others follow their strategies, and (ii) the implemented pivot variables. The welfare is non-linear in the pivot variables, but it is either monotonic in these variables and independent of (i), or is linear in (i). Moreover, both (i) and the equilibrium conditions are linear in the voting rule. This allows us to derive the linearity condition for the solution of the mechanism design problem.

Furthermore, expressing equilibria in terms of the two probabilities that a voter participates and votes $S$ or $R$ permits to solve for cost-dependent equilibrium and welfare paths when costs are small. This in turn permits a welfare comparison of specific linear mechanisms. Still, this strategy faces some difficulties because some of these paths have infinite slope - a problem that we can overcome by expressing the equilibrium paths as functions of one of the two voting probabilities. This strategy involves a considerable amount of algebra that we relegate to the appendix of the paper.

The earlier literature on costly voting in a private-values settings was largely

[5]Chakravarty et al. (2010) obtain the same conclusion for the voluntary majority rule.

concerned with particular voting rules, without asking the general design question. A number of contributions focus on the voluntary majority rule. Ledyard (1984) introduces the costly-voting setting and studies the existence and uniqueness of symmetric Bayesian equilibrium (he also endogenizes the voters' valuations by assuming that the two social alternatives arise from two candidates' political platform choices). Palfrey and Rosenthal (1983, 1985) use variants of this model to study the problem of low turnout in large elections.[6] Campbell (1999) and Taylor and Yildirim (2010) study how participation incentives differ across the two sides of the electorate in non-neutral settings. Chakravarty et al. (2010) show that introducing a voting cost is welfare-enhancing if the heterogeneity of preference intensities across the electorate is large. Krishna and Morgan (2012), in a model with a stochastic size of the electorate, show that precisely because of the presence of a participation cost, with a probability close to 1 the voluntary majority rule is approximately first-best optimal if the electorate is large.

Börgers (2004) in the neutral setting compares voluntary and compulsory majority voting and a random decision (the role of which is taken over by the constant voting rules in our model). Our results imply that the voluntary majority rule is in fact optimal among all neutral voting rules and equilibria, voluntary or compulsory. Krasa and Polborn (2009) in contrast to Börgers allow the electorate to be biased in favor of one of the social alternatives; enhancing the voluntary majority rule by an appropriate subsidy then increases welfare. Goeree and Großer (2007) study the role of a pre-election opinion poll as an enhancement of the voluntary majority rule in a setting with stochastically correlated voter types.

Little appears to be known about the Bayesian equilibria of quorum rules,[7] or of voting rules in general in the private-values costly-voting setting. Aguiar-Conraria and Magalhães (2010) provide numerical examples in which the voluntary majority rule is compared with both an approval quorum and a participation quorum. A participation quorum rule can have an equilibrium in which both sides of the electorate participate and still the status quo wins with smaller probability than it would win without a participation quorum. We advance the understanding of voting rules in the private-values costly-voting setting by providing generally applicable methods to characterize the equilibria if the participation cost is small.

Bognar et al. (2015), and Kartal (2014) appear to be the first to consider optimal-design problems in the context of costly voting. Bognar et al. (2015)

[6] For discussions of the turnout problem in common-value settings see, e.g., Feddersen and Sandroni (2006), Ghosal and Lockwood (2009), and Myatt (2014). In such models, voters may abstain purely for informational reasons (Feddersen and Pesendorfer, 1996).

[7] See Charlety, Fagart and Suam (2015) for a deterministic quorum voting model.

present a sequential voting rule that yields the first best in their setting. Kartal (2014) studies the class of neutral and monotonic voluntary voting rules in a non-neutral preference environment. In this model, the voluntary majority can be dominated by the proportional power sharing rule.

Laruelle and Valenciano (2011) introduce “weighted quaternary voting rules” which correspond to the linear rules in our model. The purpose of that paper is a taxonomy of parliamentary voting rules. The paper gives various examples of one-sided default rules applied in practice.

In the absence of voting costs, welfare-maximizing voting rules over two alternatives are well understood (e.g., Barbera and Jackson (2006), Schmitz and Tröger (2012), Azrieli and Kim (2014)). Recently, a setting with more than two alternatives has been analyzed using implementation in dominant strategies (Gershkov at al. (2014)).

The rest of the paper organized as follows. In Section 2 we introduce the model, and show that any symmetric simultaneous-move transfer-free rule is interim payoff-equivalent to a voting rule. In Section 3 we establish the optimality of linear rules and discuss quorum rules. In Section 4 we characterize the equilibria of linear rules if the participation cost is small. The formula for the welfare achieved in an optimal voting rule is stated in Section 5; various properties of optimal voting rules are derived. Section 6 deals with the neutral model. Most proofs are relegated to the Appendix.

# 2 Model

Consider $n \geq 2$ individuals who have to implement one of two possible social alternatives, denoted $S$ (“status quo”) and $R$ (“reform”). The decision about $R$ versus $S$ is made via a mechanism which determines a social alternative depending on the participating individuals’ actions. Each individual may abstain from the mechanism. We build a model that is symmetric across individuals.

**Preferences**

Each individual $i$ cares about four outcomes $R$, $iR$, $S$, and $iS$, where $R$ means that $R$ is implemented and individual $i$ abstains from participating in the mechanism, $iR$ means that $R$ is implemented and individual $i$ participates in the mechanism, and similarly for $S$ and $iS$. For each individual $i$, a state realizes. All individuals have von-Neumann-Morgenstern preferences over mappings from states into lotteries over outcomes. States are distributed identically and stochastically independently across individuals.

We assume that the preferences satisfy the following properties. First, there exists a state in which $R$ is strictly preferred to $S$. Second, there exists a state in which $S$ is strictly preferred to $R$. The properties so far serve only to exclude trivial cases. Third, in each state, $R$ is strictly preferred to $iR$ and $S$ is strictly preferred to $iS$. This means that any benefits from the act of voting are dominated by the cost of participation. Fourth, we assume for all individuals $i$ in all states the following indifference,

$$\begin{pmatrix} R & iS \\ 1/2 & 1/2 \end{pmatrix} \sim \begin{pmatrix} iR & S \\ 1/2 & 1/2 \end{pmatrix}. \tag{1}$$

In words: if each social alternative, $R$ or $S$, is implemented with probability $1/2$ and individual $i$ participates with probability $1/2$, then she does not care about the relation between her participation and the implemented social alternative.

Without loss of generality, states and preferences can be represented in the following form (see Lemma A in the Appendix). For each individual $i$, a state is a number $v_i \in \mathbb{R}$. States are distributed according to some c.d.f. $F$. An alternative is a mapping

$$v_i \mapsto \begin{pmatrix} R & iR & S & iS \\ p_R(v_i) & p_{iR}(v_i) & p_S(v_i) & p_{iS}(v_i) \end{pmatrix},$$

where $p_x(v_i)$ denotes the probability that the outcome $x$ arises, in state $v_i$. Individual $i$'s (ex-ante) expected utility is

$$\int_{\mathbb{R}} ((p_R(v_i) + p_{iR}(v_i))v_i - (p_{iR}(v_i) + p_{iS}(v_i))c)\, \mathrm{d}F(v_i), \tag{2}$$

where the parameter $c > 0$ is called the *participation cost*. Due to (1), the participation cost is independent of the implemented social alternative. The utility from the social alternative $S$ is set to 0, and individual $i$ obtains the "benefit" $v_i$ from the social alternative $R$.[8] The first of our preference properties implies $F(0) < 1$, the second implies $F(0) > 0$.

Vice versa, for any c.d.f. $F$ with $0 < F(0) < 1$ and any $c > 0$ all preference relations represented by the expected utility above satisfy our four preference properties.

[8]The earlier literature sometimes employs different preference representations, e.g., $v_i \in \{-1, 1\}$ and otherwise states are parameterized by the variable $c$, as in Palfrey and Rosenthal (1985) and Börgers (2004).

Given any state $v_i$ realized for individual $i$, she is called *type* $v_i$. We call her an $R$-*voter* if $v_i > 0$, and call her an $S$-*voter* if $v_i < 0$.

For simplicity we assume that the distribution of types, $F$, has a sufficiently smooth density $f$ with $f(0) > 0$, and $F$ has a bounded support, that is $\overline{v} \stackrel{\text{def}}{=} \sup\{v|F(v) < 1\} < \infty$ and $\underline{v} \stackrel{\text{def}}{=} \inf\{v|F(v) > 0\} > -\infty$.

### Voting mechanisms

Which rule should be employed by the individuals to determine a social alternative? In general, rules may allow individuals to abstain so that participation costs are saved.

A *(voluntary) mechanism* is a mapping $\Phi : \mathcal{A}^n \to [0, 1]$, where all individuals $i$ simultaneously select actions $a_i \in \mathcal{A}$, and reform $R$ is implemented with probability $\Phi(a_1, \ldots, a_n)$. We assume that $\mathcal{A}$ includes a particular action $A$ ("abstain").[9] From individual $i$'s point of view, the resulting lottery outcome is

$$\begin{pmatrix} R & S \\ \Phi(a_1, \ldots, a_n) & 1 - \Phi(a_1, \ldots, a_n) \end{pmatrix} \quad \text{if } a_i = A,$$

and

$$\begin{pmatrix} iR & iS \\ \Phi(a_1, \ldots, a_n) & 1 - \Phi(a_1, \ldots, a_n) \end{pmatrix} \quad \text{if } a_i \neq A.$$

Since our model is to be symmetric across individuals, we restrict attention to anonymous mechanisms, that is, we assume that $\Phi(a_{\xi(1)}, \ldots, a_{\xi(n)}) = \Phi(a_1, \ldots, a_n)$ for all action profiles $(a_1, \ldots, a_n)$ and all permutations $\xi$ of $\{1, \ldots, n\}$. This guarantees that any rule treats all individuals the same. Each individual employs a strategy $\sigma$ that specifies an action $\sigma(v_i) \in \mathcal{A}$ for each type $v_i$. If all others employ the strategy $\sigma$ and individual $i$ of type $v_i$ takes action $a_i$, then her (interim) expected utility is

$$v_i \rho^\Phi(a_i|\sigma) - c\mathbf{1}_{a_i \neq A},$$

where

$$\rho^\Phi(a_i|\sigma) = \int_{\mathbb{R}^{n-1}} \Phi(a_i, (\sigma(v_j))_{j\neq i}) \prod_{j\neq i} \mathrm{d}F(v_j)$$

denotes the probability that the social alternative $R$ is implemented, from the point of view of individual $i$.

[9] A mechanism without the action $A$ would involve compulsory participation. The analysis would be analogous to a situation with costless voting, $c = 0$, which is well-known; see Schmitz and Tröger (2012) and the references therein.

To get a fully symmetric model, we assume that individuals' behavior does not depend on their labels when playing the game, that is, we focus on *symmetric* (Bayesian) equilibria in which all individuals employ the same strategy $\sigma$, where

$$\sigma(v_i) \in \arg\max_{a_i \in A} v_i \rho^{\Phi}(a_i|\sigma) - c\mathbf{1}_{a_i \neq A} \quad \text{for all } v_i.$$

The lemma below observes that without loss of generality we may restrict attention to mechanisms with up to three actions, including $A$.[10] This holds because any participating $R$-voter (i.e., type $v_i > 0$) chooses an action that maximizes the probability of the social alternative $R$ relative to abstaining, and any participating $S$-voter chooses an action that minimizes the probability of the social alternative $R$.

**Lemma 1.** *For any anonymous mechanism $\Phi$ and symmetric Bayesian Nash equilibrium, there exists an anonymous mechanism $\Phi'$ and an (interim) payoff-equivalent Bayesian Nash equilibrium such that $\Phi'$ allows at most three actions for each voter.*

*Sketch of Proof.* Consider $\Phi$ and a symmetric Bayesian Nash equilibrium strategy $\sigma$.

An individual of type $v_i > 0$ will abstain or take an action $a_i \in \mathcal{A} \setminus \{A\}$ that maximizes $\rho^{\Phi}(a_i|\sigma)$; an individual of type $v_i < 0$ will abstain or take an action $a_i \in \mathcal{A} \setminus \{A\}$ that minimizes $\rho^{\Phi}(a_i|\sigma)$.

Consider (according to $\sigma$ and $F$) the distribution $d_>$ over actions $a_i$ across all types $v_i > 0$ that do not abstain; consider the distribution $d_<$ over actions $a_i$ across all types $v_i < 0$ that do not abstain. We obtain a new, interim payoff equivalent, equilibrium in $\Phi$ by assuming that all types $v_i > 0$ that do not abstain randomize their action according to the distribution $d_>$, and all types $v_i < 0$ that do not abstain randomize their action according to the distribution $d_<$.

Now define $\Phi'$ by restricting the set of actions to $d_>$, $d_<$, and $A$. This completes the proof.

By Lemma 1, it is sufficient to consider *voting* mechanisms (or rules) in which each individual chooses among 3 actions, denoted $A$, $S$, and $R$ (for convenience we use the same notation $S$ and $R$ as for social alternatives). Any such mechanism

[10]Thus, our analysis will be an instance of mechanism design with finite (specifically, three-elementary) action spaces. Another such exercise, in the different context of auctions and no participation cost, is Kos (2012).

can be described as a function $M : \{(r,s)|r \geq 0, s \geq 0, r+s \leq n\} \to [0,1]$, where $M(r,s) = M_{rs}$ denotes the probability that $R$ is implemented if $r$ individuals play $R$ and $s$ individuals play $S$. A special case are the *$R$-one-sided mechanisms* that are defined by the property that $M(r,s) = M(r,0)$ for all $(r,s)$, and the *$S$-one-sided mechanisms* that are defined by the property that $M(r,s) = M(0,s)$ for all $(r,s)$. Any $R$-one-sided mechanism can be represented by a simpler function $M(r)$ arising from a mechanism with only two actions, $A$ and $R$; similar for $S$-one-sided mechanisms. A voting rule that is not one-sided is called *two-sided*. A *constant* mechanism $M$ has $M(r,s) = M(0,0)$ for all $(r,s)$.

An example of a voting mechanism is the standard voluntary majority rule $M =$ vol maj, which is defined by the property $M(r,s) = 1$ if $r > s$, $M(r,s) = 1/2$ if $r = s$, $M(r,s) = 0$ otherwise.

It is sometimes useful to represent mechanisms somewhat imprecisely by ignoring the discreteness of the vote numbers $s$ and $r$, as in Figure 1.

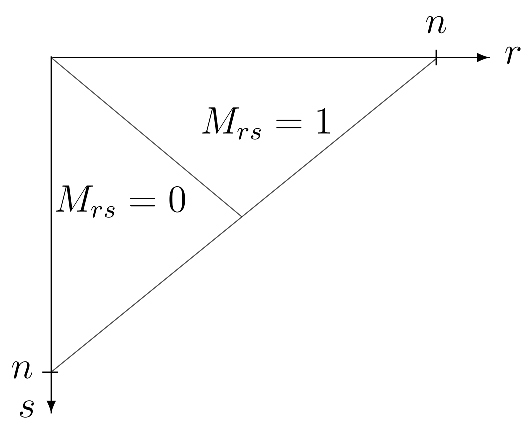


Figure 1: The voluntary majority rule $M =$ vol maj as a function of the votes $s$ and $r$.

A mechanism $M$ is called *linear* with default bias $\xi$, $R$-weight $\xi^R \geq 0$, and $S$-weight $\xi^S \geq 0$ (where $\xi^S > 0$ or $\xi^R > 0$ or $\xi \neq 0$) if, for all $s$ and $r$,

$$M_{rs} = \begin{cases} 1 & \text{if } k_{rs} > 0, \\ 0 & \text{if } k_{rs} < 0, \end{cases}$$

where we use the shortcut

$$k_{rs} = r\xi^R - s\xi^S - n\xi.$$

(Cf. Figure 2.) Observe that linearity entails no condition on $M(r, s)$ if $k_{rs} = 0$. A linear mechanism is called *upper linear* if $M(r, s) = 1$ for all $(r, s)$ with $k_{rs} = 0$. There exist only finitely many upper linear mechanisms for any given number of individuals $n$.

The class of linear rules is — up to the indeterminacy along the cutoff line where $k_{rs} = 0$ — a two-parameter class of rules because only the relative size of the three parameters $\xi$, $\xi^R$, and $\xi^S$, is relevant.

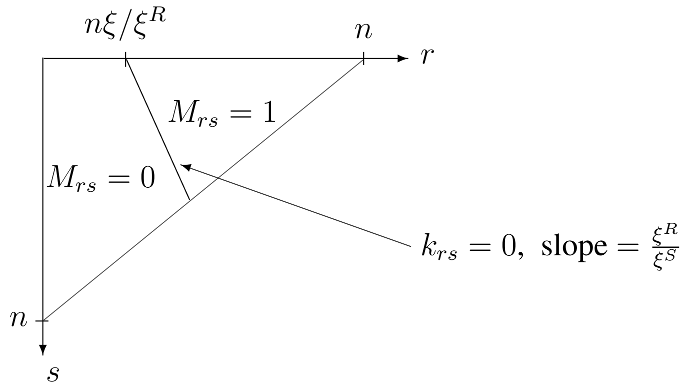


Figure 2: For each linear rule, there is a cutoff line described by the equation $k_{rs} = 0$. Above the line, alternative $R$ is chosen, below the line $S$ is chosen. If the rule is upper linear, then at each point on the cutoff line alternative $R$ is chosen.

While a linear rule can weigh votes for one alternative stronger than the votes for the other alternative (by having $\xi^R \neq \xi^S$), it can at the same time take either alternative as the "default" (by having $\xi \neq 0$) if not enough votes are received. The linear rules with default bias $\xi = 0$ are the "qualified majority rules". Another special case of linear rules are *one-sided default rules* which are defined by the property that $\xi^R = 0$ or $\xi^S = 0$; the cutoff-line $k_{rs} = 0$ in Figure 2 will be horizontal or vertical. In a one-sided default rule, at most one side of the electorate—either the $S$-voters or the $R$-voters—participate with positive probability in any equilibrium.

**Equilibria of voting mechanisms**

Consider a voting mechanism $M$ and an equilibrium strategy $\sigma$. In the following we will use the shortcuts $\rho(a_i) = \rho^M(a_i|\sigma)$ for $a_i = A, S, R$. We can assume without loss of generality that $\rho(R) \geq \rho(S)$ (relabel the actions $R$ and $S$ if necessary). Individual $i$ prefers action $R$ over action $A$ if

$$v_i(\rho(R) - \rho(A)) \;\; \geq \;\; c,$$

and prefers $S$ over $A$ if

$$v_i(\rho(S) - \rho(A)) \;\; \geq \;\; c.$$

If $\rho(R) < \rho(A)$, then no positive type will participate; we can replace the mechanism $M$ by an $S$-one-sided mechanism $\hat{M}$ (if $\rho(S) < \rho(R)$, nobody will take action $R$, otherwise $\rho(S) = \rho(R)$ and one has to argue as in the proof of Lemma 1). Hence, $\rho^{\hat{M}}(R|\sigma) = \rho^{\hat{M}}(A|\sigma)$. Thus, without loss of generality we can assume that

$$\rho(R) \geq \rho(A) \quad \text{and, similarly,} \quad \rho(S) \leq \rho(A).$$

Define thresholds

$$v_+ \;\; = \;\; \begin{cases} \frac{c}{\rho(R)-\rho(A)}, & \text{if } \rho(R) > \rho(A), \\ \infty, & \text{otherwise,} \end{cases}$$

$$v_- \;\; = \;\; \begin{cases} \frac{c}{\rho(S)-\rho(A)}, & \text{if } \rho(A) > \rho(S), \\ -\infty, & \text{otherwise.} \end{cases}$$

Then $\sigma$ is a symmetric equilibrium strategy if and only if

$$\sigma(v_i) \;\; = \;\; \begin{cases} R & \text{if } v_i > v_+, \\ A & \text{if } v_i < v_i < v_+, \\ S & \text{if } v_i < v_-, \end{cases}$$

and randomization is possible for $v_i \in \{v_+, v_-\}$.

Observe that (because $F$ has no atom) any equilibrium is fully described by the parameters $v^+$ and $v^-$. Alternatively, we can describe an equilibrium in terms of the *pivot variables*

$$\Delta^R = \rho(R) - \rho(A) \in [0,1], \quad \Delta^S = \rho(A) - \rho(S) \in [0,1]$$

that capture the impact that an $R$-vote and, resp., $S$-vote has relative to abstention. The probability that a given individual casts an $R$-vote (resp., $S$-vote) is denoted

$$\tau^R \;\; = \;\; l^R(\Delta^R) = \begin{cases} 1 - F(\frac{c}{\Delta^R}), & \text{if } \Delta^R \geq c/\overline{v}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau^S \;\; = \;\; l^S(\Delta^S) = \begin{cases} F(-\frac{c}{\Delta^S}), & \text{if } \Delta^S \geq -c/\underline{v}, \\ 0 & \text{otherwise.} \end{cases}$$

Given participation rates $\tau^R$ and $\tau^S$, the mechanism $M$ determines the impact $k^{R,M}(\tau^R, \tau^S)$ that an $R$-vote has relative to abstention and the impact $k^{S,M}(\tau^R, \tau^S)$ that an $S$-vote has relative to abstention. Using basic properties of the multinomial distribution,

$$k^{R,M}(\tau^R, \tau^S) = \sum_{r+s \le n-1} (M_{r+1,s} - M_{r,s}) \begin{pmatrix} n-1 \\ r \; s \end{pmatrix} (\tau^R)^r (\tau^S)^s (1 - \tau^R - \tau^S)^{n-1-r-s}, \tag{3}$$

$$k^{S,M}(\tau^R, \tau^S) = \sum_{r+s \le n-1} (M_{r,s} - M_{r,s+1}) \begin{pmatrix} n-1 \\ r \; s \end{pmatrix} (\tau^R)^r (\tau^S)^s (1 - \tau^R - \tau^S)^{n-1-r-s}, \tag{4}$$

where $\begin{pmatrix} n-1 \\ r \; s \end{pmatrix} = \frac{(n-1)!}{r!s!(n-1-r-s)!}$.

Given these definitions, a pair $(\Delta^R, \Delta^S)$ corresponds to an equilibrium if and only if

$$\Delta^R = k^{R,M}(l^R(\Delta^R), l^S(\Delta^S)), \tag{5}$$

$$\Delta^S = k^{S,M}(l^R(\Delta^R), l^S(\Delta^S)). \tag{6}$$

This is a continuous fixed-point equation on $[0,1]^2$. Hence, it has a fixed point by Brouwer's theorem. That is, for each mechanism $M$ there exists at least one equilibrium.

We can summarize the equilibrium conditions (5) and (6) by saying that (i) the pivot variables determine the participation rates (via the functions $l^S$ and $l^R$) and (ii) the participation rates determine the pivot variables (via the functions $k^{R,M}$ and $k^{S,M}$). Observe that an important simplification stems from the fact that the mechanism $M$ plays a role only in part (ii).

An equilibrium can also be described as a pair of participation rates $(\tau^R, \tau^S)$ satisfying the conditions

$$\tau^R = l^R(k^{R,M}(\tau^R, \tau^S)), \tag{7}$$

$$\tau^S = l^S(k^{S,M}(\tau^R, \tau^S)). \tag{8}$$

### Optimal mechanisms

Any mechanism-equilibrium pair $m = (M, \Delta^R, \Delta^S)$ (or $m = (M, \tau^R, \tau^S)$) defines, for each type $v_i$, an interim expected utility

$$U^m(v_i) = v_i \rho^M(A) + \begin{cases} \max\{v_i \Delta^R - c, 0\} & \text{if } v_i > 0, \\ \max\{-v_i \Delta^S - c, 0\} & \text{if } v_i < 0, \end{cases}$$

where (with $\tau^R = l^R(\Delta^R)$ and $\tau^S = l^S(\Delta^S)$)

$$\rho^M(A) \quad = \sum_{r+s \leq n-1} M_{rs} \left( \begin{array}{c} n-1 \\ r \;\; s \end{array} \right) (\tau^R)^r (\tau^S)^s (1-\tau^R-\tau^S)^{n-1-r-s}. \quad (9)$$

Our analysis covers welfare maximization both from an ex-ante and an interim point of view. A social planner who is interested in the *ex-ante* expected welfare of each individual implements a mechanism-equilibrium pair $m = (M, \Delta^R, \Delta^S)$ that maximizes (10) below with $g(v_i) = 1$ for all $v_i$.

From an *interim* point of view, a social planner can be interested in a larger class of mechanism-equilibrium pairs. Different types $v_i$ in general prefer different mechanism-selection pairs. The crucial question then is whether a mechanism-equilibrium pair is immune against an interim Pareto improvement across all types. Consider any $m$ that maximize *g-welfare*

$$\begin{aligned} W_g(m) \stackrel{\text{def}}{=} E_F[g(v_i)U^m(v_i)] \quad &= \quad E_F[g(v_i)v_i]\rho^M(A) + \int_0^{\overline{v}} \max\{v_i\Delta^R - c, 0\} g(v_i) \mathrm{d}F(v_i) \\ &+ \int_{\underline{v}}^0 \max\{-v_i\Delta^S - c, 0\} g(v_i) \mathrm{d}F(v_i) \qquad (10) \end{aligned}$$

for some "welfare weights" $g(v_i) \geq 0$ for all $v_i$ with $\int g(v_i)\mathrm{d}F(v_i) = 1$. Any such $m$ is weakly interim efficient: no other mechanism-equilibrium pair can make all types strictly better off.[11] We say that $g$ puts positive weight on all types if $g(v_i) > 0$ for all $v_i$ in the support of $F$.

Observe that the $g$-welfare directly depends on $M$ only via its dependence on $\rho^M(A)$. On the other hand, $\rho^M(A)$ depends on $\Delta^R$ and $\Delta^S$.

We say that a mechanism-equilibrium pair $m^*$ is *optimal with welfare weights* $g$ if $m^*$ solves the problem

$$\begin{array}{lll} (g\text{-opt}) & \max\limits_{(M,\Delta^R,\Delta^S)} & W_g(M, \Delta^R, \Delta^S) \\ & \text{s.t.} & (5),\ (6), \\ & & 0 \leq M_{rs} \leq 1 \quad \text{for all } (r, s). \end{array}$$

[11]Vice versa, any interim-efficient mechanism-equilibrium pair is $g$-optimal for some $g$ if we allow for a public randomization device. Formally, consider the set $\mathcal{U}$ of utility vectors $U^m$, where $m$ varies across all mechanism-equilibrium pairs. Let $\overline{\mathcal{U}}$ denote the convex hull of $\mathcal{U}$; any point in $\overline{\mathcal{U}}$ can be realized by using a public randomization device to select a mechanism-equilibrium pair. We may call $U \in \overline{\mathcal{U}}$ interim-incentive-efficient if for all $V \in \overline{\mathcal{U}} \setminus \{U\}$ we have $V(v_i) < U(v_i)$ for some $v_i$. By the separating-hyperplane theorem, any interim-incentive-efficient $m$ is $g$-optimal for some $g$.

Implicit to our formulation is the classical mechanism-design doctrine: the designer selects an equilibrium with highest welfare if the optimal mechanism has multiple equilibria.

It is useful to have shortcuts for the $g$-weighted conditional expected preference intensity on the two sides of the electorate,

$$E_- = E[-v_i g(v_i)|v_i < 0], \quad E_+ = E[v_i g(v_i)|v_i > 0].$$

If $E_- = 0$, then the constant rule $M_{rs} = 1$ together with the equilibrium $(0, 0)$ solves ($g$-opt); if $E_+ = 0$, the constant rule $M_{rs} = 0$ is optimal. To exclude these trivial cases, we assume that

$$E_- > 0 \quad \text{and} \quad E_+ > 0. \tag{11}$$

The constraint set of problem ($g$-opt) is bounded and, by a standard continuity argument, closed. Hence, a $g$-optimum exists by Weierstraß' Maximum-Value Theorem. However, in general the objective of problem ($g$-opt) is *not* quasi-concave in $(\Delta^R, \Delta^S)$.[12]

# 3 Optimality of linear rules

Our first main contribution is that the search for optimal mechanisms can be restricted to (upper) linear mechanisms. In particular, ex-ante optimality of a mechanism-equilibrium pair implies that it is outcome-equivalent to a pair in which the mechanism is linear.

**Proposition 1.** *1. For any welfare weights $g$, there exists a $g$-optimal mechanism-equilibrium pair in which the mechanism is upper linear.*

*2. Consider any $g$-optimal mechanism-equilibrium pair $(M, \tau^{R*}, \tau^{S*})$ and assume that $g$ puts positive weight on all types or $E[v_i g(v_i)] \neq 0$. If $\tau^{R*} > 0$ and $\tau^{S*} > 0$, then $M$ is linear. If $\tau^{R*} > 0$ and $\tau^{S*} = 0$, then $M(r, 0)$ is as in an $R$-one*

[12]This can be seen most easily for welfare weights $g$ such that $E[v_i g(v_i)] = 0$. For such $g$, $W_g$ is independent of $M$ and in the range where $\Delta^R > \frac{c}{\overline{v}}$ and $\Delta^S > \frac{c}{-\underline{v}}$, the derivatives with respect to the pivot variables are strictly increasing in $\Delta^R$ and $\Delta^S$, respectively, because

$$\frac{\partial W_g}{\partial \Delta^R} = \int_{[\frac{c}{\Delta^R}, \overline{v}]} v g(v) \mathrm{d}F(v), \quad \frac{\partial W_g}{\partial \Delta^S} = -\int_{[\underline{v}, -\frac{c}{\Delta^S}]} v g(v) \mathrm{d}F(v).$$

Thus, in the range considered, $W_g$ is strictly convex in $(\Delta^R, \Delta^S)$.

*sided default rule. If $\tau^{R*} = 0$ and $\tau^{S*} > 0$, then $M(0, s)$ is as in an $S$-one sided default rule.*

Here is a sketch of the proof (details can be found in the Appendix). To prove the first part, we fix a $g$-optimal mechanism-equilibrium pair and consider the set $\mathcal{M}$ of all mechanisms in which this particular equilibrium occurs. The set $\mathcal{M}$ is a convex polyhedron because the equilibrium conditions are linear in the mechanism. Moreover, the $g$-welfare is maximized by either maximizing or minimizing the linear objective $\rho^M(A)$ across all $M \in \mathcal{M}$. Thus, the optimum is characterized by the Kuhn-Tucker first-order conditions, which yield shadow prices (Lagrange parameters) for relaxing the two constraints (5) and (6). The value of the objective $\rho^M(A)$ minus the total cost of relaxing the constraints is a linear function of $M$. If in this linear function the coefficient of $M_{rs}$ is positive, then $M_{rs} = 1$, and $M_{rs} = 0$ if the coefficient is negative. From (3), (4), and (9), the coefficient is a linear combination of the probabilities of the events $(r, s)$, $(r, s-1)$, and $(r-1, s)$ that refer to three possible action profiles of the $n-1$ other voters from a given voters' point of view. Due to the stochastic independence of the voters' types, the probability of the event $(r, s)$ times $s(1 - \tau^R - \tau^S)$ equals the probability of the event $(r, s-1)$ times $(n-r-s)\tau^S$. One sees that this relation is linear in $r$ and $s$, and there is a similar linear relation between $(r, s)$ and $(r-1, s)$. Thus, the sign of the coefficient is linear in $r$ and $s$.

To prove the second part, consider the case $E[v_i g(v_i)] \neq 0$. Then the proof of the first part shows the existence of the parameters $\xi$, $\xi^R$, and $\xi^S$ such that $M$ is linear.

The case in which $E[v_i g(v_i)] = 0$ is more complex. We consider a relaxed problem in which the equilibrium conditions (5) and (6) are only required as inequalities. Because $E[v_i g(v_i)] = 0$, the $g$-welfare is independent of $M$. Using this we show that any $g$-optimal mechanism-equilibrium pair also solves the relaxed problem. For any fixed pair of participation rates, the projection of the feasible set of the relaxed problem onto the space of mechanisms is a convex set. This set is mapped onto a convex set in $\mathbb{R}^2$ by the pair of right-hand-sides of (5) and (6). Any optimal mechanism is mapped to a point on the weak Pareto frontier of that set (otherwise one could change the mechanism so that (5) and (6) become strict, and could then increase $\Delta^R$ and $\Delta^S$ so that the objective obtains a higher value). At any point on the weak Pareto frontier the separating hyperplane theorem can be applied. This yields a necessary condition for optimality that implies that the optimal mechanism is linear.

**Quorum rules**

Some of the most commonly used voting rules require not only a qualified majority in order to validate a reform, but also require that a certain number of voters participates or approves the reform. Formally, given parameters $q$ with $0 < q < 1$ ("quorum") and $\alpha > 0$, a voting rule $M$ is an *approval quorum rule* with parameters $(q, \alpha)$ if $M_{rs} = 1$ for all $(r, s)$ with $r \geq \max\{qn, \alpha s\}$ and $M_{rs} = 0$ otherwise. A voting rule $M$ is a *participation quorum rule* with parameters $(q, \alpha)$ if $M_{rs} = 1$ for all $(r, s)$ with $r \geq \max\{qn - s, \alpha s\}$ and $M_{rs} = 0$ otherwise. Any approval quorum rule with parameters $(q, \alpha)$, $\alpha > q/(n - q)$, is non-linear if $n$ is sufficiently large; this is due to the kink in the cutoff line between the regions in the $(r, s)$-space where $M_{rs} = 1$ and where $M_{rs} = 0$. Any participation quorum rule with parameters $(q, \alpha)$ is non-linear if $n$ is sufficiently large because it is non-monotonic, that is, an additional $S$-vote can reduce the probability that $R$ is implemented.

The common use of quorum rules may be due to an attempt to find a compromise between two goals, to save voting costs and to keep the spirit of a majority decision. But this attempt misses the strategic incentives of the voters.

Quorum rules have been criticized because they have equilibria $(\tau^R, \tau^S)$ with $\tau^R = 0$ which may be interpreted as a "misrepresentation of preferences" (e.g., Corte-Real and Pereira, 2004). Proposition 1 shows that a quorum rule can be optimal *only if* such an equilibrium is played or the rule is in fact linear. No non-linear quorum rule together with an equilibrium in which both sides of the electorate participate can be optimal.[13] We will show below that in many relevant environments optimality requires that both sides participate (see Section 5 below). In such environments all equilibria of all non-linear quorum rules are sub-optimal.

# 4 Equilibria of linear rules with small participation costs

In this section, we provide generally applicable methods for characterizing the equilibria of voting rules if the participation cost is small. The results of this section are used in Section 5 to find welfare-maximizing rules. Proposition 1 permits us to focus attention on upper linear rules.

[13] Towards characterizing the equilibria of quorum rules, observe that Lemma 3 extends verbatim to any quorum rule that is identical to the considered upper-linear rule at all $(r, s)$ with $r + s \geq n - k - 1$.

Given any non-constant upper linear rule $M$, the threshold of $R$-votes needed to implement $R$ if all $S$-voters abstain is denoted

$$r^* \quad = \quad \arg\min\{r \mid M_{r,0} = 1\}.$$

(Observe that $r^*$ is well-defined because $M_{n,0} = 1$ for any non-constant upper linear rule.) Similarly, define

$$s^* \quad = \quad \arg\min\{s \mid M_{0,s} = 0\}.$$

From upper linearity, $r^* = 0$ or $s^* = 0$, but not both. We call $S$ the default alternative if $s^* = 0$; a minimum of $r^* \geq 1$ $R$-votes is then needed to implement the alternative $R$. Vice versa, $R$ is the default alternative if $r^* = 0$. The minimum number of $R$-votes needed to implement $R$ if everybody participates is denoted

$$t^* \quad = \quad \arg\min\{r \mid M_{r,n-r} = 1\}.$$

Observe that $t^* \geq 1$ because $M$ is non-constant. For all $r = 0, 1, \ldots, n$, define

$$\overline{y}(r) \quad = \quad rE_+ - (n-r)E_-.$$

Here, $\overline{y}(r)$ can be interpreted as the aggregate welfare, conditional on the event that $r$ individuals prefer $R$, $n-r$ individuals prefer $S$, and $R$ is implemented. For all $z = 0, 1, \ldots, n$, define

$$w_g(z) \quad = \quad \frac{1}{n}\sum_{r=z}^{n}\binom{n}{r}(1-F(0))^r F(0)^{n-r}\overline{y}(r). \qquad (12)$$

This is the $g$-welfare that arises in the 0-participation-cost limit assuming that $R$ is implemented if and only if at least $z$ individuals prefer $R$.

By distinguishing six categories of mechanism-equilibrium pairs, the lemma below describes a necessary condition for mechanism-equilibrium pairs when the participation cost is small. In category (i), $M$ is two-sided and almost everybody participates. In (ii), $r^* \geq 1$, most of the $R$-voters participate, and at most a few of the $S$-voters do. In (iii), $r^* = 0$ and at most a few of the $S$-voters participate. Categories (iv) and (v) are like (ii) and (iii) with the roles of $R$ and $S$ exchanged. In (vi), at most a few individuals participates.

The mechanism-equilibrium pairs in categories (iii), (v), and (vi) will not be relevant for welfare-maximization (cf. Remark 1 below). In category (iii), with small $c > 0$ the $R$-voters may participate at any rate, but their pivot variable will be close to 0 so that the welfare of a constant rule is approximated. The same is true in category (v). Obviously the welfare of a constant rule is also approximated in category (vi). Thus, the relevant mechanism-equilibrium categories are (i), (ii), and (iv); the 0-participation cost limit of the welfare achieved in these categories is described in the lemma.

**Lemma 2.** *Consider any non-constant upper linear rule $M$. Let $\epsilon > 0$. Then, for all $c > 0$ sufficiently close to 0, the following holds for all mechanism-equilibrium pairs $(M, \tau^R, \tau^S)$ with participation cost $c$:*

$$
\begin{array}{rl}
(i) & M \text{ two-sided}, \ (\tau^R, \tau^S) \in (1 - F(0) - \epsilon, 1 - F(0)) \times (F(0) - \epsilon, F(0)), \\
\text{or } (ii) & r^* \geq 1, \ M_{r^*,1} = 1, \ (\tau^R, \tau^S) \in (1 - F(0) - \epsilon, 1 - F(0)) \times [0, \epsilon), \\
\text{or } (iii) & r^* = 0, \ (\tau^R, \tau^S) \in [0, 1 - F(0)) \times [0, \epsilon), \\
\text{or } (iv) & s^* \geq 1, \ M_{1,s^*} = 0, \ (\tau^R, \tau^S) \in [0, \epsilon) \times (F(0) - \epsilon, F(0)), \\
\text{or } (v) & s^* = 0, \ (\tau^R, \tau^S) \in [0, \epsilon) \times [0, 1 - F(0)), \\
\text{or } (vi) & (\tau^R, \tau^S) \in [0, \epsilon) \times [0, \epsilon).
\end{array}
$$

*As $c \to 0$, the welfare $W_g(M, \tau^R, \tau^S)$ converges to $w_g(t^*)$ for all mechanism-equilibrium pairs in category (i). The limit welfare in category (ii) is $w_g(r^*)$, and is $w_g(n - s^* + 1)$ in category (iv). The welfare converges to $E[vg(v)]M_{00}$ for all mechanism-equilibrium pairs in categories (iii), (v), and (vi).*

To prove this lemma (details are in the Appendix), we consider a converging sequence of equilibria as the participation cost tends to 0. If the corresponding sequence of $R$-pivot variables converges to a number $> 0$, then every $R$-voter participates in the limit as the cost vanishes, similar so for the $S$-pivot variable. A careful case distinction then yields the categories of mechanism-equilibrium pairs described in the lemma. In particular, if the limit of the participation rates is positive for both sides of the electorate, then the rule is two-sided and *everybody* participates in the limit (category (i)); otherwise every election result $(r, s)$ would have a positive probability in the limit, implying (by two-sidedness) a strictly positive limit $R$-pivot variable and thus full participation of the $R$-voters in the limit and similarly for the $S$-voters, a contradiction.

**Remark 1.** *If $c$ is small, then any $g$-optimal rule is non-constant.*

To see this, observe first that, from Lemma 3 below, an equilibrium as in (i) exists for every two-sided $M$. By setting $t^* = 1$, one can achieve a welfare of $w_g(1) > E[vg(v)]$ (the inequality is strict due to (11)), and by setting $t^* = n$

one achieves a welfare $w_g(n) > 0$ (again using (11)) to get strictness). Thus, the maximum welfare is greater than $\max\{0, E[vg(v)]\}$, which equals the maximum welfare achievable by a constant rule.

Remark 1 implies that the mechanism-equilibrium pairs of the categories (iii), (v), and (vi) are no candidates for optimality if $c$ is close to $0$.

In the following, we will characterize the mechanism-equilibrium pairs described in (i), (ii), and (iv).

### Mechanism-equilibrium pairs in category (i)

Consider a mechanism-equilibrium pair in category (i). Because $M$ is two-sided, there exists a minimal $k$ such that

$$M(t^* - 1, n - t^* + 1 - k) = 1.$$

Let

$$\hat{k} \;\; = \;\; n - t^* + 1 - k \geq 0.$$

Here is a sketch of some entries $M_{rs}$ with $r$ on the horizontal and $s$ on the vertical axis. The parameter $k$ is equal to the number of 0s below row $s = \hat{k}$ in column $r = t^* - 1$.

$$\begin{array}{lcccc}
\hat{k} & \cdot & \mathbf{1} & 1 & 1 \\
 & 0 & 0 & 1 & 1 \\
 & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 1 & 1 \\
n - t^* & 0 & 0 & 1 & \\
 & 0 & 0 & & \\
 & 0 & & & \\
 & & & t^* &
\end{array}$$

Similarly, there exists a minimal $\check{k}$ such that

$$M(t^* - \check{k}, n - t^*) \;\; = \;\; 0.$$

Observe that $k \geq 2$ or $\check{k} \geq 2$, but not both (because $M_{t^*-1,n-t^*} = 0$ or $= 1$). Also, if $k \geq 2$ then $t^* < n$, and if $\check{k} \geq 2$ then $t^* > 1$.

We consider the case $k \geq 2$ first. Note that $\hat{k} = n - t^* - 1$ if $k = 2$. It is useful to introduce the shortcut

$$\Delta^{t*} \quad = \quad \binom{n-1}{t^*-1} (1 - F(0))^{t^*-1} F(0)^{n-t^*}.$$

Taking the point of view of any voter, $\Delta^{t*}$ equals the probability that exactly $t^* - 1$ of the other individuals prefer $R$.

The following lemma says that, for small cost levels, any two-sided upper linear rule with $k \geq 2$ has a unique equilibrium in category (i), that is, in which almost everybody participates (an analogous result can be stated if $\check{k} \geq 2$). In the limit $c \to 0$, the $R$-pivot probability tends to $\Delta^{t*}$, and the $S$-pivot probability vanishes (see (16)).

There is a smooth relationship between cost and equilibrium participation levels. The shape of this relationship can be described explicitly, in terms of $k$th order derivatives (15) while all lower-order derivatives vanish (14). Understanding this shape is a crucial precondition for computing the welfare effect (17).

The divergence in (13) is a major obstacle towards understanding the existence and properties of the equilibria. We overcome this obstacle by a trick. We take $\tau^S$ as the independent variable and describe $c$ and $\tau^R$ as functions of $\tau^S$.

The constructed equilibrium in which almost everybody participates can be seen as a perturbation of the full-participation equilibrium of $M$ if $c = 0$. By using $\tau^S$ as the independent variable, we perturb the equilibrium at $c = 0$ at an infinitely smaller rate than how the perturbation would be done with $c$ as the independent variable. Thus we restore differentiability.

Considering the functions $\tau^R(c)$ and $\tau^S(c)$, introducing a small $c$ lowers the participation of the $R$-voters at rate $f(0)/\Delta^{t*}$ (see (13)), which is independent of $k$. The participation rate of the $S$-voters is lowered at an initially infinite rate. The higher $k$, the more often we have to differentiate $c(\tau^S)$ at $\tau^S = F(0)$ until the derivative becomes non-zero. Thus, the larger $k$, the less steep is the function $c(\tau^S)$ at $\tau^S = F(0)$. Thus, the larger $k$, the faster is the drop of the inverse $\tau^S(c)$ at small $c$. Put differently, the larger $k$, the smaller the participation rate of the $S$-voters at any given small $c$.

Formula (17) describes the first-order welfare effect of introducing a participation cost. The effect consists of a term proportional to $f(0)$ that captures the welfare effect arising from types close to 0 beginning to abstain, and the term $-1$ that captures that there is essentially full participation so that all voters have to pay the participation cost.

**Lemma 3.** *Consider any two-sided upper linear rule $M$ with $k \geq 2$. Then there exist $\overline{\epsilon} > 0$ and $\overline{c} > 0$ with the following properties.*

*For all $c \in (0, \overline{c})$ there exists a unique equilibrium $(\tau^R(c), \tau^S(c))$ in $(1 - F(0) - \overline{\epsilon}, 1 - F(0)) \times (F(0) - \overline{\epsilon}, F(0))$.*

*The functions $\tau^R(c)$ and $\tau^S(c)$ are strictly decreasing and continuous on $(0, \overline{c})$ and extend continuously to 0 with $\tau^R(0) = 1 - F(0)$ and $\tau^S(0) = F(0)$. The derivatives*

$$(\tau^R)'(0) = -\frac{f(0)}{\Delta^{t*}}, \quad \lim_{c \to 0, c \neq 0} (\tau^S)'(c) = -\infty. \tag{13}$$

*The inverse functions $c(\tau^S)$ and $\tau^R(\tau^S) = \tau^R(c(\tau^S))$ are $k$ times continuously differentiable at $\tau^S = F(0)$ with*

$$\frac{d^l c}{d(\tau^S)^l}|_{\tau^S = F(0)} = 0, \quad \frac{d^l \tau^R}{d(\tau^S)^l}|_{\tau^S = F(0)} = 0 \;\; \textit{for all } 1 \leq l < k \tag{14}$$

*and*

$$\left( \begin{array}{c} \frac{d^k c}{d(\tau^S)^k}\Big|_{\tau^S = F(0)} \\ \frac{d^k \tau^R}{d(\tau^S)^k}\Big|_{\tau^S = F(0)} \end{array} \right) = \frac{(n-1)!}{(t^* - 1)! \hat{k}!} (1 - F(0))^{t^* - 1} F(0)^{\hat{k}} (-1)^{k-1} \cdot k \cdot \left( \begin{array}{c} -\frac{1}{f(0)} \\ \frac{1}{\Delta^{t*}} \end{array} \right) \tag{15}$$

*Also*

$$k^{R,M}(1 - F(0), F(0)) = \Delta^{t*}, \quad k^{S,M}(1 - F(0), F(0)) = 0. \tag{16}$$

*Letting $W(c) = W_g(M, \tau^R(c), \tau^S(c))$,*

$$W'(0) \;\; = \;\; -f(0) \left(1 - \frac{1}{k}\right) \overline{y}(t^* - 1) - 1. \tag{17}$$

Here is a sketch of the proof (details are in the Appendix). Given any mechanism, the equilibrium conditions (7) and (8) can be written as equations in $c$, $\tau^S$, and $\tau^R$ such that these equations extend smoothly to the full-participation point $(c, \tau^R, \tau^S) = (0, 1 - F(0), F(0))$. The direct approach of representing $\tau^S$ and $\tau^R$ as differentiable functions of $c$ in a right-neighborhood of $c = 0$ via the implicit function theorem does not work because $\tau^S(c)$ turns out to be non-differentiable at $c = 0$ (see (13)). We overcome this problem as follows. We apply the implicit function theorem using $\tau^S$ as the independent variable and thus obtain functions $c(\tau^S)$ and $\tau^R(\tau^S)$ in a left-neighborhood of $\tau^S = F(0)$. Computing the first $k$

derivatives of these functions (see (14) and (15)) we see that $c$ is strictly decreasing in $\tau^S$ and $\tau^R$ is strictly increasing. Hence, $c(\tau^S) > 0$ and $\tau^R(\tau^S) < 1 - F(0)$ so that the pair $(\tau^R(\tau^S), \tau^S)$ is an equilibrium at cost level $c(\tau^S)$.

Formulas (16) are immediate from the definition of $M$.

To prove (17), we represent the welfare as a function $\hat{W}(\tau^S)$ in a left-neighborhood of the point $\tau^S = F(0)$. The first $k-1$ derivatives of $\hat{W}$ vanish at 0. Thus, we can apply L'Hospital's rule together with (15) to find the desired first-order effect. While (17) is simple, its proof relies on extensive algebra.

The next case to consider is as in Lemma 3, with condition $k \geq 2$ replaced by the condition $\check{k} \geq 2$.

**Lemma 4.** *Consider any two-sided upper linear rule $M$ that satisfies $\check{k} \geq 2$. Then there exist $\overline{\epsilon} > 0$ and $\overline{c} > 0$ with the following properties. For all $c \in (0, \overline{c})$ there exists a unique equilibrium $(\tau^R(c), \tau^S(c))$ in $(1 - F(0) - \overline{\epsilon}, 1 - F(0)) \times (F(0) - \overline{\epsilon}, F(0))$.*

*Letting $W(c) = W_g(M, \tau^R(c), \tau^S(c))$, we have*

$$W'(0) \quad = \quad f(0)\left(1 - \frac{1}{\check{k}}\right)\overline{y}(t^*) - 1. \tag{18}$$

*Proof.* Beginning with Lemma 3, do the following replacements of parameters: $k \to \check{k}$, $t^* \to n - t^* + 1$, and $F \to -F$.

### Mechanism-equilibrium pairs in categories (ii) and (iv)

Consider a mechanism-equilibrium pair in category (ii) (an analogous analysis applies to category (iv)). We begin with the case in which $M$ is two-sided. Thus, there exists a minimal $q \geq 2$ such that

$$M(r^*, q) = 0.$$

Here is a sketch of some entries $M_{rs}$ with $r$ on the horizontal and $s$ on the vertical axis. The parameter $q$ is equal to the number of 1s in column $r^*$.

$$\begin{array}{c c c c c c c}
0 & \quad & \cdot & 0 & \mathbf{1} & 1 & 1 \\
1 & & 0 & 0 & \mathbf{1} & 1 & 1 \\
 & & \vdots & \vdots & \vdots & \vdots & \vdots \\
q-1 & & 0 & 0 & \mathbf{1} & 1 & 1 \\
q & & 0 & 0 & 0 & 1 & \\
 & & 0 & 0 & 0 & & \\
 & & 0 & 0 & & & \\
 & & 0 & & r^* & & 
\end{array}$$

The following lemma says that, for small cost levels, any two-sided upper linear rule satisfying $r^* \geq 1$ and $M_{r^*,1} = 1$ has a unique equilibrium in category (ii) in which the $S$-voters do not completely abstain (i.e., $\tau^S(c) > 0$). Again, there is a smooth relationship between cost and participation levels in this equilibrium that can be described explicitly, this time in terms of $(q-1)$th order derivatives (21) while all lower-order derivatives vanish (20). Again, there is a divergence problem (19) that is overcome by taking $\tau^S$ as the independent variable and describing $c$ and $\tau^R$ as functions of $\tau^S$.

Formula (22) describes the first-order welfare effect of introducing a participation cost. The effect consists of a term proportional to $f(0)$ that captures the welfare effect arising from types close to 0 beginning to abstain, and the term $\int_0^{\overline{v}} g(v)\mathrm{d}F(v)$ that captures that almost all of the $R$-voters participate and have to pay the participation cost. The first-order effect is independent of $q$.

**Lemma 5.** *Consider any two-sided upper linear rule $M$ with $r^* \geq 1$ and $M_{r^*,1} = 1$. Then there exist $\overline{\epsilon} > 0$ and $\overline{c} > 0$ such that the following properties hold.*

*For all $c \in (0, \overline{c})$, there exists a unique equilibrium $(\tau^R(c), \tau^S(c))$ in $(1 - F(0) - \overline{\epsilon}, 1 - F(0)) \times (0, \overline{\epsilon})$.*

*The functions $\tau^R(c)$ and $\tau^S(c)$ are continuous and extend continuously to 0 with $\tau^R(0) = 1 - F(0)$ and $\tau^S(0) = 0$. The function $\tau^R$ is strictly decreasing and $\tau^S$ is strictly increasing. The derivatives*

$$(\tau^R)'(0) = -\frac{f(0)}{\Delta^{r*}}, \quad \lim_{c\to 0, c\neq 0} (\tau^S)'(c) = \infty. \tag{19}$$

*The inverse functions $c(\tau^S)$ and $\tau^R(\tau^S) = \tau^R(c(\tau^S))$ are $q-1$ times continuously differentiable at $\tau^S = 0$ with*

$$\frac{d^l c}{d(\tau^S)^l}|_{\tau^S = F(0)} = 0, \quad \frac{d^l \tau^R}{d(\tau^S)^l}|_{\tau^S = F(0)} = 0 \ \ \textit{for all } l = 1, \dots, q-2 \tag{20}$$

*and*

$$\left( \begin{array}{c} \frac{d^{q-1} c}{d(\tau^S)^{q-1}}\Big|_{\tau^S=0} \\ \frac{d^{q-1} \tau^R}{d(\tau^S)^{q-1}}\Big|_{\tau^S=0} \end{array} \right) = \frac{-\underline{v}(n-1)!}{r^*!(n-r^*-q)!}(1-F(0))^{r^*} F(0)^{n-q-r^*} \left( \begin{array}{c} 1 \\ -\frac{f(0)}{\Delta^{r*}} \end{array} \right). \tag{21}$$

*Also $k^R(1 - F(0), 0) = \Delta^{r*}$ and $k^S(1 - F(0), 0) = 0$.*

*Letting $W(c) = W_g(M, \tau^R(c), \tau^S(c))$, we have*

$$W'(0) \;=\; f(0)\left(E_+ - \overline{y}(r^*)\right) - \int_0^{\overline{v}} g(v)dF(v). \tag{22}$$

The main steps of the proof are analogous to the proof of Lemma 3, except that now we can stop already at the $(q-1)$th derivatives rather than at the $k$th derivatives. A sketch is in the Appendix.

Consider now category (ii) when $M$ is a one-sided upper linear rule, or is a two-sided upper linear rule in which we look for an equilibrium in which the $S$-voters abstain completely. That is, $\tau^S = 0$.

**Lemma 6.** *Consider any non-constant upper linear rule $M$ with $r^* \geq 1$ and $M_{r^*,1} = 1$. Then there exists $\overline{\epsilon} > 0$ and $\overline{c} > 0$ such that the following holds. For all $c \in [0, \overline{c})$ there exists $\tau^R(c)$ such that $(\tau^R(c), 0)$ is the unique equilibrium in $(1 - F(0) - \overline{\epsilon}, 1 - F(0)) \times \{0\}$ if $c > 0$.*

*The function $\tau^R(c)$ is continuous and strictly decreasing on $[0, \overline{c})$ with $\tau^R(0) = 1 - F(0)$. The derivative*

$$(\tau^R)'(0) \;=\; -\frac{f(0)}{\Delta^{r*}}.$$

*Also $k^R(1 - F(0), 0) = \Delta^{r*}$ and $k^S(1 - F(0), 0) = 0$.*

*Letting $W(c) = W_g(M, \tau^R(c), 0)$, formula (22) holds.*

The proof applies the implicit function theorem to the equilibrium condition (7). Details are in the Appendix. The first-order welfare effect is the same as in Lemma 5. This is natural: in equilibrium, very likely at most one $S$-voter participates and in this case the one-sided rule is identical to the two-sided rule considered in Lemma 5.

Finally, consider category (iv). Because everything is analogous to category (ii), we keep the exposition short. The lemma below is the mirror of Lemma 5 and Lemma 6. To prove it, one applies formula (22) with the replacements $r^* \to s^*$ and $F \to -F$.

**Lemma 7.** *Consider any non-constant upper linear rule $M$ that satisfies $s^* \geq 1$ and $M_{1,s^*} = 0$. Then there exists $\overline{\epsilon} > 0$ and $\overline{c} > 0$ such that, for all $c \in [0, \overline{c})$ the following holds.*

*There exists $\tau^S(c)$ such that $(0, \tau^S(c))$ is the unique equilibrium in $\{0\} \times (F(0) - \overline{\epsilon}, F(0))$ if $c > 0$. The function $\tau^S(c)$ is continuous on $[0, \overline{c})$. Letting $W(c) = W_g(M, 0, \tau^S(c))$, we have*

$$W'(0) \;\; = \;\; f(0)\left(E_- + \overline{y}(n - s^*)\right) - \int_{\underline{v}}^{0} g(v)dF(v). \tag{23}$$

*Now suppose that $M$ is two-sided. There exists $\tau^R(c)$ and $\tau^S(c)$ such that $(\tau^R(c), \tau^S(c))$ is the unique equilibrium in $(0, \overline{\epsilon}) \times (F(0) - \overline{\epsilon}, F(0))$ if $c > 0$. The functions $\tau^R(c)$ and $\tau^S(c)$ are continuous on $[0, \overline{c})$ with $\tau^R(0) = 0$ and $\tau^S(0) = F(0)$. Letting $W(c) = W_g(M, \tau^R(c), \tau^S(c))$, (23) holds.*

The results in this section provide a complete characterization of the relevant equilibria of upper linear voting rules if the participation cost is small. We can make the following observations concerning equilibrium multiplicity. Any non-constant one-sided upper linear rule has a single equilibrium beyond any possible equilibria that create welfare levels close to 0. Now consider a two-sided rule. Recall that $r^* \geq 1$ or $s^* \geq 1$, but not both. Without loss of generality, assume $r^* \geq 1$. Beyond any possible equilibria that create welfare levels close to 0, the rule has three equilibria if $M_{r^*,1} = 1$, and has one equilibrium otherwise: the equilibrium with almost full participation described in Lemma 3 with some $t^* > r^*$ always exists, and the other two equilibria (described in Lemma 5 and Lemma 6 in which at most a few $S$-voters participate) exist if and only if $M_{r^*,1} = 1$.

# 5 Optimal voting rules

We are now equipped to determine $g$-optimal voting rules if the participation cost is small. In order to achieve a welfare comparison of the finitely many upper linear rules if the participation cost is small, it is sufficient to consider those mechanism-equilibrium pairs that are optimal in the 0-participation-cost limit and compare these according to the first-order welfare effects of introducing a small cost.

Let $k^*$ denote the smallest number of $R$-voters such that the social alternative $R$ is welfare maximizing, that is, $k^*$ is minimal with the property $\overline{y}(k^*) \geq 0$. From (11), $k^* \geq 1$. Let $k^{**} = k^*$ unless $\overline{y}(k^*) = 0$, in which case $k^{**} = k^* + 1$. That is, $k^{**}$ is the smallest number of $R$-voters such that the social alternative $S$ is not welfare maximizing.

It is well-known (e.g., Barbera and Jackson, 2006) that in the 0-participation cost limit, the maximum $g$-welfare among all mechanism-equilibrium pairs is

$w_g(k^*)$ $(= w_g(k^{**}))$. Thus, using Lemma 2, if the participation cost is small, then any optimal mechanism-equilibrium pair in which the mechanism is upper linear belongs to category (i) with $t^* = k^*$ or $t^* = k^{**}$, to category (ii) with $r^* = k^*$ or $r^* = k^{**}$, or to category (iv) with $s^* = n+1-k^*$ or $s^* = n+1-k^{**}$. We have to compare these with respect to the first-order welfare effects of introducing a small cost. Define

$$W'_{RS0} = f(0)\left(-\frac{(k^*-1)(n-k^*)}{n-k^*+1}E_+ + (n-k^*)E_-\right) - 1, \quad (24)$$

$$W'_{RS1} = f(0)\left(-\frac{(n-k^{**})(k^{**}-1)}{k^{**}}E_- + (k^{**}-1)E_+\right) - 1, \quad (25)$$

$$W'_R = f(0)\left((n-k^*)E_- - (k^*-1)E_+\right) - \int_0^{\overline{v}} g(v)\mathrm{d}F(v), \quad (26)$$

$$W'_S = f(0)\left((k^{**}-1)E_+ - (n-k^{**})E_-\right) - \int_{\underline{v}}^0 g(v)\mathrm{d}F(v). \quad (27)$$

Let

$$\mathcal{W} = \{W'_{RS0}, W'_{RS1}, W'_R, W'_S\},$$
$$W'_* = \max \mathcal{W},$$
$$\mathcal{W}_* = \{w \in \mathcal{W} \mid w = W'_*\}.$$

The following proposition is our main characterization result concerning $g$-optimality if the participation cost is small. There is an explicit formula (28) for the first-order welfare effect of introducing a small participation cost.[14] We distinguish four cases depending on which of the four numbers defined above is the largest.[15] In case 1, the optimal mechanism is *almost-one sided*; the single entry $M_{k^*-1,0} = 1$ makes it two-sided; every voter participates with a high probability. Case 2 is analogous to case 1; here, the entry $M_{0,n-k^{**}} = 0$ prevents $M$ from being one-sided. In case 3, the first two rows of $M$ are as in a one-sided mechanism, every $R$-voter participates with a high probability, and every $S$-voter participates with a low probability. Case 4 is analogous to case 3, with reversed roles of $R$ and $S$. Observe that a qualified majority rule in which each voter participates with a probability close to 1 is not consistent with any of the four cases unless $n = 2$.

[14] $\mathbf{o}(c)$ denotes a function with $\lim_{c\to 0} \mathbf{o}(c)/c = 0$.

[15] No statements are made concerning the non-generic cases in which the maximum is not unique; we would need to compare higher-order welfare effects to fill these gaps.

**Proposition 2.** *The solution value to problem (g-opt) as a function of the participation cost* $c$ *is*

$$W(c) \quad = \quad w_g(k^*) + c \cdot W'_* + \mathbf{o}(c). \tag{28}$$

*Let* $\epsilon > 0$. *There exists* $\overline{c} > 0$ *such that, for all* $c \in (0, \overline{c})$, *all* $g$*-optimal mechanism-equilibrium pairs* $(M, \tau^R, \tau^S)$ *in which* $M$ *is upper linear have the following properties.*

1. *If* $\mathcal{W}_* = \{W'_{RS0}\}$, *then* $k^* < n$, $t^* = k^*$, $k = n - k^* + 1$, $\tau^R > 1 - F(0) - \epsilon$, *and* $\tau^S > F(0) - \epsilon$;
2. *If* $\mathcal{W}_* = \{W'_{RS1}\}$, *then* $k^{**} > 1$, $t^* = k^{**}$, $\check{k} = k^{**}$, $\tau^R > 1 - F(0) - \epsilon$, *and* $\tau^S > F(0) - \epsilon$;
3. *If* $\mathcal{W}_* = \{W'_R\}$, *then* $r^* = k^*$, $M_{r^*,1} = 1$, $\tau^R > 1 - F(0) - \epsilon$, *and* $\tau^S < \epsilon$;
4. *If* $\mathcal{W}_* = \{W'_S\}$, *then* $s^* = n + 1 - k^{**}$, $M_{1,s^*} = 0$, $\tau^R < \epsilon$, *and* $\tau^S > F(0) - \epsilon$.

*Proof.* To verify (28), we have to show that $W'_*$ is the largest first-order welfare effect across all upper-linear mechanism-equilibrium pairs that are welfare-maximizing in the 0-cost limit.

Consider the mechanism-equilibrium pairs in category (i). Among these, a necessary condition for maximizing the $g$-welfare is optimality in the 0-participation cost limit, implying $t^* = k^*$ or $t^* = k^{**}$. If $k \geq 2$, then maximizing the first-order effect (17) requires that one chooses $k$ as a large as possible, that is, $k = n - t^* - 1$ (that is, $\hat{k} = 0$). This yields the first-order effect (24) if $t^* = k^*$. Observe that $W'_{RS0} > -1$ because $\overline{y}(k^* - 1) < 0$. In the case $k^{**} \neq k^*$, setting $t^* = k^{**}$ would yield the first-order effect $-1$ (cf. (17) and observe that $\overline{y}(k^{**} - 1) = \overline{y}(k^*) = 0$).

Similarly, if $\check{k} \geq 2$, then maximizing the first-order effect (18) requires that one chooses $t^* = k^{**}$ and to choose $\check{k}$ as a large as possible, that is, $\check{k} = k^{**}$. This yields the first-order effect (25).

Among the mechanism-equilibrium pairs in category (ii), optimality in the 0-participation cost limit implies $r^* = k^*$ or $r^* = k^{**}$. The larger first-order effect (22) is obtained at $r^* = k^*$, yielding (26). A similar argument (using (23)) applied to the mechanism-equilibrium pairs in category (iv) yields (27).

Thus, it remains to compare the first-order effects $W'_{RS0}$, $W'_{RS1}$, $W'_R$, and $W'_S$. The comparison yields the four cases of the proposition. This completes the proof.

To conclude our welfare analysis, we consider *compulsory voting*. That is, we consider an optimal mechanism without action $A$. The first-order welfare effect is then equal to $-1$. This is never optimal.

**Remark 2.** *Voluntary voting dominates compulsory voting if the participation cost is small. Formally,* $W'_* > -1$.

*Proof.*

$$W'_{RS0} + W'_{RS1} \geq f(0) \underbrace{\left( \frac{k^* - 1}{n - k^* + 1} E_+ + \frac{n - k^*}{k^*} E_- \right)}_{>0} - 2. \tag{29}$$

(The inequality is an equality $k^{**} = k^*$.)

It is instructive to contrast this result with Krasa and Polborn (2009). They consider the standard majority rule with costly participation and show that one can increase the welfare by introducing a voting subsidy. A subsidy can be interpreted as an incentive to vote; a high subsidy leads to compulsory voting. While compulsory voting may dominate voluntary voting for a fixed voting rule, our results show that voluntary voting dominates if an optimal voting rule is used.

We conclude with three comparative-statics results concerning the density $f(0)$ of voter types that are near indifferent between the social alternatives. Given any welfare weights $g$, a valid thought experiment consists in changing the type distribution $F$ by changing $f(0)$, while keeping $F(0)$, $E_+$, and $E_-$ fixed. Here, $f(0)$ may be chosen arbitrarily large or small. Increasing $f(0)$ means that for some types $v_i > 0$ the density $f(v_i)$ is also increased, as is the density $f(v_i)$ for some $v_i < 0$. In this sense, a large $f(0)$ capture an electorate that is rather heterogeneous with respect to preference intensities: ex-ante, each voter is rather uncertain whether or not she will care about the social alternative. Vice versa, a small $f(0)$ is consistent with an electorate that is homogeneous with respect to preference intensities (for example, $F$ may approximate a two-point distribution).

**Remark 3.** *If the population features sufficiently heterogeneous preference intensities, then essentially everybody on both sides of the electorate should participate; in the opposite case of homogeneous preference intensities, essentially only one side should participate. Formally, there exists $f_0^*$ such that the following holds: If $f(0) > (<) f_0^*$, then* $\max\{W'_{RS0}, W'_{RS1}\} > (<) \max\{W'_R, W'_S\}$.

*Proof.* Without loss of generality, consider the case $W'_{RS0} \geq W'_{RS1}$. Note that the sign of $W'_{RS0} - W'_{RS1}$ is independent of $f(0)$ because

$$\begin{aligned} & W'_{RS0} - W'_{RS1} \\ & = f(0) \underbrace{\left( \left( n - k^* + \frac{(n-k^{**})(k^{**}-1)}{k^{**}} \right) E_- - \left( k^{**} - 1 + \frac{(k^*-1)(n-k^*)}{n-k^*+1} \right) E_+ \right)}_{\geq 0}. \end{aligned} \tag{30}$$

Suppose that $W'_{RS0} \geq \max\{W'_R, W'_S\}$ for some $f(0)$. It is sufficient to show that the inequality becomes strict if we increase $f(0)$. Observe that

$$W'_{RS0} - W'_R = f(0) \frac{k^* - 1}{n - k^* + 1} E_+ - \int_{\underline{v}}^{0} g(v) \mathrm{d}F(v) \geq 0.$$

In particular, $k^* > 1$. Thus, the difference is strictly increasing in $f(0)$.

Observe that

$$\begin{aligned} & W'_{RS0} - W'_S \\ & = f(0) \underbrace{\left( -\frac{(k^*-1)(n-k^*)}{n-k^*+1} E_+ + (n-k^*)E_- - (k^{**}-1)E_+ + (n-k^{**})E_- \right)}_{(*)} \\ & \qquad - \int_0^{\overline{v}} g(v) \mathrm{d}F(v). \end{aligned}$$

Thus, $k^* < n$ (otherwise $k^{**} = n$, thus $W'_{RS0} - W'_S < 0$). Hence, $(*) > 0$ from (30)). Thus, the difference $W'_{RS0} - W'_S$ is strictly increasing in $f(0)$. This completes the proof.

The second comparative-statics result states that increasing voting costs artificially can be welfare improving; while this would not necessarily be surprising for a sub-optimal voting rule, here we obtain this result for rules that are welfare-maximizing given the participation cost level.

**Remark 4.** *If the preference intensities are sufficiently heterogeneous, then it is optimal to artificially increase the voting cost in any environment with a sufficiently small voting cost. Formally, if $f(0)$ is sufficiently large, then $W'_* > 0$.*

*Proof. Due to (29), with large $f(0)$ at least one of the effects, $W'_{RS0}$ or $W'_{RS1}$, is $> 0$.*

The third comparative-statics result asks which side of the electorate should participate.

**Remark 5.** *Consider cases in which the electorate is rather homogeneous with respect to preference intensities so that only one side of the electorate should optimally participate; it should be the "minority" if there is a sufficiently strong asymmetry across the two social alternatives in terms of preferences or welfare weights.*

*Formally: If $f(0)$ is small and $\int_{\underline{v}}^{0} g(v)dF(v) \gg 1/2$, then $W_R' > W_S'$; if $f(0)$ is small and $\int_{\underline{v}}^{0} g(v)dF(v) \ll 1/2$, then $W_R' < W_S'$.*

# 6 Neutral environments and voting rules

In some applications the alternatives $R$ and $S$ cannot by objectively categorized (as, say, status quo and reform), but are fundamentally symmetric, like two new candidates running for an office; in such applications, only rules that are neutral in the sense of treating both alternatives identically may be considered legitimate. Formally, a voting rule $M$ is *neutral* if

$$M(r,s) \;=\; 1 - M(s,r) \quad \text{for all } s \text{ and } r.$$

In this section, we analyze neutral rules in a *neutral environment*, that is, an environment in which the distribution $F$ is symmetric around 0; in particular, then, $\overline{v} = -\underline{v}$. The results here hold for arbitrary participation costs.

Any neutral rule in a neutral environment has a neutral equilibrium $(\Delta^R, \Delta^S)$ in the sense that

$$\Delta^R = \Delta^S \stackrel{\text{def}}{=} \Delta.$$

We focus on such equilibria. In particular, then,

$$\rho^M(A) \;=\; \frac{1}{2}.$$

Dropping the constant term $E_g[v_i]/2$, the expression of the welfare simplifies to

$$W_g(\Delta) \;=\; \int_0^{\overline{v}} \max\{v_i\Delta - c, 0\} g(v_i) \mathrm{d}F(v_i) + \int_{\underline{v}}^{0} \max\{-v_i\Delta - c, 0\} g(v_i) \mathrm{d}F(v_i). \tag{31}$$

Thus, the problem boils down to maximizing $\Delta$.[16]

Here is the main result in this section.[17] The result holds for arbitrary participation costs.

**Proposition 3.** *In any neutral environment, for arbitrary welfare weights $g$, the voluntary majority rule maximizes $g$-welfare across all neutral equilibria of all neutral voting rules.*

*Sketch of proof.* We want to show that the standard (voluntary) majority rule has the largest $\Delta$.

To see this, denote $l = l^R = l^S$ and $\tau = l(\Delta)$ the probability that a voter chooses $R$ (or $S$). We have

$$\Delta = h^M(\tau) \overset{\text{def}}{=} \sum_{r+s \leq n-1} (M(r+1,s) - M(r,s)) \begin{pmatrix} n-1 \\ r \;\; s \end{pmatrix} \tau^{r+s}(1-2\tau)^{n-1-r-s}.$$

Observe that the function $h^M$ is non-linear and generally non-monotonic.

The equilibrium conditions (5) and (6) boil down to twice the same condition:

$$h^M(l(\Delta)) = \Delta.$$

In general, this equation may have multiple solutions corresponding to multiple equilibria.

We will show that, for all neutral $M$,

$$h^{\text{vol maj}}(\tau) \quad \geq \quad h^M(\tau) \quad \text{for all } \tau. \tag{32}$$

Because $l$ is increasing, this implies the desired optimality of the standard majority rule, that is,

$$\Delta^{\text{vol maj}} \quad \geq \quad \Delta^M;$$

Figure 3 should make this clear.

[16] It is interesting to compare this with Börger's (2004) insight that too many voters participate in a voluntary majority rule compared to the participation rate selected by a social planner. We find that an optimal rule is such that *as many voters as possible* participate, but our statement is subject to already having taken into account the voters' incentives.

[17] While this result may appear un-surprising, other neutral settings are known in which the standard majority rule is suboptimal, even among neutral rules and equilibria: Schmitz and Tröger (2012) show that, in the absence of voting costs, but with correlated valuations, a weak majority rule can yield a higher welfare. In a weak majority rule, a lottery is used if neither alternative has a sufficiently strong majority over the other alternative.

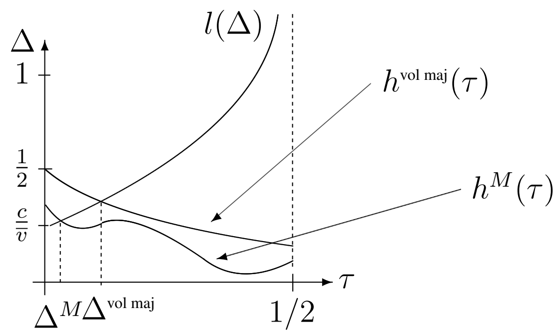


Figure 3: The intersection points of $l$ with the $h$-functions indicates the equilibrium points. The standard majority rule is optimal because the equilibria of any neutral mechanism are to the left of its equilibrium.

We can show (32) as follows.[18] Consider any $(r', s')$ with $r' > s'$. The derivative of $h^M(\tau)$ with respect to $M(r', s')$, taking into account the constraint $M(r', s') = 1 - M(r', s')$, equals

$$\begin{aligned}
& \binom{n-1}{r'-1\ \ s'} \tau^{r'-1+s'}(1-2\tau)^{n-1-(r'-1+s')} - \binom{n-1}{r'\ \ s'} \tau^{r'+s'}(1-2\tau)^{n-1-(r'+s')} \\
& - \binom{n-1}{s'-1\ \ r'} \tau^{s'-1+r'}(1-2\tau)^{n-1-(s'-1+r')} + \binom{n-1}{s'\ \ r'} \tau^{s'+r'}(1-2\tau)^{n-1-(s'+r')} \\
= \ & \tau^{s'+r'-1}(1-2\tau)^{n-1-(s'+r')} \\
& \cdot \left( \binom{n-1}{r'-1\ \ s'} (1-2\tau) - \binom{n-1}{r'\ \ s'} \tau - \binom{n-1}{s'-1\ \ r'} (1-2\tau) + \binom{n-1}{s'\ \ r'} \tau \right) \\
= \ & \tau^{s'+r'-1}(1-2\tau)^{n-(s'+r')} \\
& \qquad \cdot \frac{(n-1)!}{(r'-1)!(s'-1)!(n-1-(r'+s'-1))!} \left( \frac{1}{s'} - \frac{1}{r'} \right) \\
> \ & 0.
\end{aligned}$$

Hence, it is optimal to choose $M(r', s')$ as large as possible, that is, $= 1$. This proves (32) and thus proves the optimality of the standard majority rule. This completes the proof.

[18] In the special case $\tau = 1$, the essence of the arguments towards showing (32) goes back to Rae (1969) and is similar to arguments in Schmitz and Tröger (2012).

We may also consider neutral rules with compulsory participation, that is, rules without action $A$.

**Corollary 1.** *In any neutral environment, the voluntary majority rule maximizes the ex-ante welfare across all neutral equilibria of neutral voting rules including rules with compulsory participation.*

To see this, note that, by the results of Schmitz and Tröger (2012), the majority rule is optimal among all rules with compulsory participation (note that voting costs play no role if only compulsory rules are considered, cf. footnote 9). Moreover, Börgers' (2004) analysis implies that, in terms of ex-ante expected utility, the voluntary majority rule is better than the compulsory majority rule.

# 7 Appendix

**Lemma A.** *An individual's preferences over state-dependent lotteries are represented by the utility function (2).*

*Proof.* We can take the state space to be some $\Omega \subseteq \mathbb{R}^4$; this allows to assign a Bernoulli utility to each of the four outcomes in each state. Given any individual $i$, an alternative $\lambda$ for individual $i$ is a mapping from $\Omega$ into the lotteries over the four outcomes $R$, $iR$, $S$, and $iS$. The lottery assigned in state $\omega \in \Omega$ is denoted

$$\lambda_\omega = \begin{pmatrix} R & iR & S & iS \\ \pi_R^\omega & \pi_{iR}^\omega & \pi_S^\omega & \pi_{iS}^\omega \end{pmatrix},$$

where $\pi_x^\omega$ denotes the probability that the outcome $x$ arises, in state $\omega$.

By assumption, the preferences over state-dependent lotteries $\lambda$ have a Bernoulli utility representation. That is, the expected utility can be written as

$$\int_\Omega (u_R^\omega \pi_R^\omega + u_{iR}^\omega \pi_{iR}^\omega + u_S^\omega \pi_S^\omega + u_{iS}^\omega \pi_{iS}^\omega)\, \mu(\mathrm{d}\omega),$$

where $u_x^\omega$ denotes the Bernoulli utility from outcome $x$ in state $\omega$, and $\mu$ denotes the measure on the set of states. From (1),

$$u_R^\omega \frac{1}{2} + u_{iS}^\omega \frac{1}{2} \quad = \quad u_{iR}^\omega \frac{1}{2} + u_S^\omega \frac{1}{2}.$$

Thus, using the shortcut $c^\omega = u_R^\omega - u_{iR}^\omega$,

$$u_{iR}^\omega = u_R^\omega - c^\omega, \quad u_{iS}^\omega = u_S^\omega - c^\omega.$$

Observe that $c^\omega > 0$ by the third of our preference properties.

The expected utility can also be written as

$$\begin{aligned} & \int_\Omega u_S^\omega \mu(\mathrm{d}\omega) + \int_\Omega \left( \left( u_R^\omega - u_S^\omega \right) \left( \pi_R^\omega + \pi_{iR}^\omega \right) - \left( \pi_{iR}^\omega + \pi_{iS}^\omega \right) c^\omega \right) \mu(\mathrm{d}\omega) \\ = \quad & \int_\Omega u_S^\omega \mu(\mathrm{d}\omega) + \int_\Omega \left( \frac{u_R^\omega - u_S^\omega}{c^\omega} \left( \pi_R^\omega + \pi_{iR}^\omega \right) - \left( \pi_{iR}^\omega + \pi_{iS}^\omega \right) \right) c^\omega \mu(\mathrm{d}\omega) \end{aligned}$$

Define

$$v^\omega := \frac{u_R^\omega - u_S^\omega}{c^\omega}$$

and define the probability measure $\mu'$ via $\mu'(B) = \int_B c^\omega \mu(\mathrm{d}\omega) / \int_\Omega c^\omega \mu(\mathrm{d}\omega)$ for all measurable sets $B \subseteq \Omega$. Then, dropping the first term, the expected utility above can be written as

$$\int_\Omega \left( v^\omega \left( \pi_R^\omega + \pi_{iR}^\omega \right) - \left( \pi_{iR}^\omega + \pi_{iS}^\omega \right) c \right) \mu'(\mathrm{d}\omega),$$

where $c = 1$.

Thus, the preferences in any state are fully determined by $v^\omega$. We summarize all states $\omega$ with the same $v^\omega$ into a single state $v_i$; the lotteries assigned to different identified $\omega$s are replaced by their $\mu'$-compound lottery. Formally, the random variable $Z : \omega \mapsto v^\omega$ is distributed according to the c.d.f.

$$F(v_i) \quad = \quad \int_{\omega,\ v^\omega \le v_i} \mu'(\mathrm{d}\omega).$$

There exists a conditional probability measure $\mu'(\cdot | Z = v_i)$ on $\Omega$. For all outcomes $x$, let

$$p_x(v_i) \quad = \quad \int_\Omega \pi_x^\omega \mu'(\mathrm{d}\omega | Z = v_i).$$

Using Fubini's theorem for transition probabilities (e.g., Bauer, Probability Theory, Chapter 36),

$$\begin{aligned} & \int_\Omega \left( v^\omega \left( \pi_R^\omega + \pi_{iR}^\omega \right) - \left( \pi_{iR}^\omega + \pi_{iS}^\omega \right) c \right) \mu'(\mathrm{d}\omega) \\ = \quad & \int_{\mathbb{R}} \int_\Omega \left( v_i \left( \pi_R^\omega + \pi_{iR}^\omega \right) - \left( \pi_{iR}^\omega + \pi_{iS}^\omega \right) c \right) \mu'(\mathrm{d}\omega | Z = v_i) \mathrm{d}F(v_i). \\ = \quad & \int_{\mathbb{R}} \left( (p_R(v_i) + p_{iR}(v_i)) v_i - (p_{iR}(v_i) + p_{iS}(v_i)) \cdot c \right) \mathrm{d}F(v_i). \end{aligned}$$

This completes the proof.

*Proof of Proposition 1.* Without loss of generality, assume that $E[v_i g(v_i)] \geq 0$.

To prove the first part, consider a $g$-optimal mechanism-equilibrium pair $(M^*, \Delta^{R*}, \Delta^{S*})$. Let $\tau^{R*}$ and $\tau^{S*}$ denote the corresponding participation rates.

Suppose first that $\tau^{R*} > 0$ and $\tau^{S*} > 0$.

Define the (convex) set of mechanisms

$$\begin{aligned}\mathcal{M} \;=\; \{M \mid \; & \Delta^{R*} = k^{R,M}(\tau^{R*}, \tau^{S*}), \\ & \Delta^{S*} = k^{S,M}(\tau^{R*}, \tau^{S*}), \\ & 0 \leq M_{rs} \leq 1 \quad \text{for all } (r,s)\}.\end{aligned}$$

For any $M \in \mathcal{M}$, the pair $(\Delta^{R*}, \Delta^{S*})$ is an equilibrium. Thus, by $g$-optimality, the mechanism $M = M^*$ solves the following problem:

$$\begin{aligned}\max_{M} \quad & E[v_i g(v_i)]\rho^M(A) \\ \text{s.t.} \quad & M \in \mathcal{M}.\end{aligned}$$

Because $E[v_i g(v_i)] \geq 0$, any solution $M = M^{**}$ to the following problem also yields a $g$-optimal $(M^{**}, \Delta^{R*}, \Delta^{S*})$:

$$\begin{aligned}\text{(lin)} \quad \max_{M} \quad & \rho^M(A) \\ \text{s.t.} \quad & M \in \mathcal{M}.\end{aligned}$$

Problem (lin) is linear. Hence, the Kuhn-Tucker conditions are necessary and sufficient for this problem, without any constraint qualification. Thus, there exist Lagrange multipliers[19] $\mu^R$, $\mu^S$, and $\mu_{r,s}$ for all $(r,s)$ such that

$$\mu_{r,s} \;=\; \frac{\partial}{\partial M_{rs}} \left( \rho(A) + \mu^R k^{R,M^{**}}(\tau^{R*}, \tau^{S*}) + \mu^S k^{S,M^{**}}(\tau^{R*}, \tau^{S*}) \right), \quad (33)$$

where $\mu_{r,s} \leq 0$ if $M^{**}_{rs} < 1$ and $\mu_{r,s} \geq 0$ if $M^{**}_{rs} > 0$ (complementary slackness).

Suppose first that $r > 0$, $s > 0$, and $r + s \leq n - 1$. Then (33) implies

$$\mu_{r,s} \;=\; \begin{pmatrix} n-1 \\ r \;\; s \end{pmatrix} (\tau^{R*})^{r-1} (\tau^{S*})^{s-1} (1 - \tau^{R*} - \tau^{S*})^{n-1-r-s} \frac{k_{rs}}{n-r-s},$$

[19] We do not include the constraints $0 \leq M_{rs} \leq 1$ when defining the Lagrangian, but maximize the Lagrangian subject to these constraints. Thus, there are not Lagrange multipliers for these constraints.

where we use the shortcut

$$k_{rs} = r\mu^R\tau^{S*}(1-\tau^{R*}-\tau^{S*}) + (n-r-s)(1-\mu^R-\mu^S)\tau^{R*}\tau^{S*} + s\mu^S\tau^{R*}(1-\tau^{R*}-\tau^{S*}).$$

Using the complementary slackness conditions, we find a solution

$$M^{**}_{rs} = \begin{cases} 1 & \text{if } k_{rs} \geq 0, \\ 0 & \text{if } k_{rs} < 0. \end{cases}$$

(It is easy to verify the points $(r, s)$ with $r = 0$, $s = 0$, or $r + s = n - 1$ as well.)

Define

$$\begin{aligned} \xi^R &= \mu^R\tau^{S*}(1-\tau^{R*}-\tau^{S*}) - (1-\mu^R-\mu^S)\tau^{R*}\tau^{S*}, \\ \xi &= (1-\mu^R-\mu^S)\tau^{R*}\tau^{S*}, \\ \xi^S &= \mu^S\tau^{R*}(1-\tau^{R*}-\tau^{S*}) - (1-\mu^R-\mu^S)\tau^{R*}\tau^{S*}. \end{aligned}$$

Observe that $\xi^R \geq 0$ (otherwise $M^{**}_{r+1,s} \leq M^{**}_{rs}$ for all $(r, s)$, implying $\Delta^{R*} = 0$ and hence $\tau^{R*} = 0$). Similarly, $\xi^S \geq 0$.

Moreover, if $\xi = 0$, then $1 - \mu^R - \mu^S = 0$, implying $\mu^R > 0$ or $\mu^S > 0$, thus $\xi^R > 0$ or $\xi^S > 0$.

Hence, $M^{**}$ is an upper linear mechanism.

Now suppose that $\tau^{R*} > 0$ and $\tau^{S*} = 0$ (the case $\tau^{R*} = 0$ and $\tau^{S*} > 0$ is analogous and omitted).

Define the (convex) set of mechanisms

$$\begin{aligned} \mathcal{M}^R = \{M \mid & \Delta^{R*} = k^{R,M}(\tau^{R*}, 0), \\ & M_{rs} = M_{r0} \quad \text{for all } (r, s), \\ & 0 \leq M_{rs} \leq 1 \quad \text{for all } (r, s)\}. \end{aligned}$$

For any $M \in \mathcal{M}^R$, the pair $(\Delta^{R*}, 0)$ is an equilibrium. Thus, by $g$-optimality, the mechanism $M = M^*$ solves the following problem:

$$\begin{aligned} \max_M \quad & E[v_i g(v_i)]\rho^M(A) \\ \text{s.t.} \quad & M \in \mathcal{M}^R. \end{aligned}$$

Because $E[v_i g(v_i)] \geq 0$, any solution $M = M^{**}$ to the following problem also yields a $g$-optimal $(M^{**}, \Delta^{R*}, \Delta^{S*})$:

$$\text{(lin)}^R \quad \begin{aligned} \max_M \quad & \rho^M(A) \\ \text{s.t.} \quad & M \in \mathcal{M}^R. \end{aligned}$$

Problem (lin)$^R$ is linear. Hence, the Kuhn-Tucker conditions are necessary and sufficient for this problem, without any constraint qualification. Thus, there exist Lagrange multipliers $\mu^R$ and $\mu_r$ for all $r$ such that

$$\mu_r = \frac{\partial}{\partial M_{r0}} \left(\rho(A) + \mu^R k^{R,M^{**}}(\tau^{R*}, 0)\right), \tag{34}$$

where $\mu_r \leq 0$ if $M^{**}_{r0} < 1$ and $\mu_r \geq 0$ if $M^{**}_{r0} > 0$ (complementary slackness).

Suppose first that $r > 0$ and $r \leq n-1$. Then (34) implies

$$\mu_r = \frac{(n-1)!}{(n-r)!r!}(\tau^{R*})^{r-1}(1-\tau^{R*})^{n-1-r}k_r,$$

where we use the shortcut

$$k_r = (n-r)\tau^{R*}(1-\mu^R) + r(1-\tau^{R*})\mu^R.$$

Using the complementary slackness conditions, we find a solution

$$M^{**}_{rs} = \begin{cases} 1 & \text{if } k_r \geq 0, \\ 0 & \text{if } k_r < 0. \end{cases}$$

Define

$$\begin{aligned} \xi^R &= (1-\tau^{R*})\mu^R - \tau^{R*}(1-\mu^R), \\ \xi &= \tau^{R*}(1-\mu^R), \\ \xi^S &= 0. \end{aligned}$$

Observe that $\xi^R \geq 0$ (otherwise $M^{**}_{r+1,s} \leq M^{**}_{rs}$ for all $(r,s)$, implying $\Delta^{R*} = 0$ and hence $\tau^{R*} = 0$).

Moreover, if $\xi = 0$, then $1 - \mu^R = 0$, implying $\xi^R > 0$.

Hence, $M^{**}$ is an upper linear mechanism.

To prove the second part, suppose first that $E[v_i g(v_i)] \neq 0$. Then the arguments in the proof of the first part imply the desired claims.

Suppose that $E[v_i g(v_i)] = 0$. Then $W_g(M, \Delta^R, \Delta^S) \stackrel{\text{def}}{=} W(\Delta^R, \Delta^S)$ is independent of $M$.

Consider a relaxed maximization problem in which the equilibrium conditions are replaced by inequalities:

$$\begin{aligned} \text{(relax)} \quad \max_{(M,\Delta^R,\Delta^S)} \quad & W(\Delta^R, \Delta^S) \\ \text{s.t.} \quad & \Delta^S - k^{S,M}(\tau^R, \tau^S) \leq 0, \quad (S) \\ & \Delta^R - k^{R,M}(\tau^R, \tau^S) \leq 0, \quad (R) \\ & 0 \leq M_{rs} \leq 1 \quad \text{for all } (r,s). \end{aligned}$$

where we use the shortcuts $\tau^S = l^S(\Delta^S)$ and $\tau^R = l^R(\Delta^R)$.

The relaxed problem is much easier to deal with than the original problem of finding an optimal voting rule. Lemma B below justifies our focus on the relaxed problem.

**Lemma B.** *Problem (relax) always has a solution such that both $(S)$ and $(R)$ are satisfied with equality. In particular, if $E[v_i g(v_i)] = 0$, then any $g$-optimal mechanism-equilibrium pair solves problem (relax).*

The proof of Lemma B explores the linearity of the constraints in $M$, the monotonicity of the objective $W$ in the pivot variables, and the local optimality of a solution. We distinguish the cases 1, optimality of no participation at all, 2, optimality of one-sided participation, and 3, optimality of two-sided participation.

Observe first that Problem (relax) always has a solution. This follows from Weierstrass' Maximum-value Theorem.

Now consider any solution $(M, \Delta^R, \Delta^S)$ to the relaxed problem. We will construct from it another solution to the same problem such that the constraints $(S)$ and $(R)$ are satisfied with equality.

Let $\tau^R$ and $\tau^S$ be the participation rates corresponding to $\Delta^R$ and $\Delta^S$. We will distinguish three cases.

*Case 1.* Suppose that $\tau^S = 0$ and $\tau^R = 0$. Then another solution of the relaxed problem is given by $(\hat{M}, 0, 0)$ with $\hat{M}_{rs} =$ const for all $s$ and $r$. The solution $(\hat{M}, 0, 0)$ satisfies both constraints $(S)$ and $(R)$ with equality, so that it is a solution of the original problem.

*Case 2.* Suppose that $\tau^S = 0$ and $\tau^R > 0$ (the case $\tau^R = 0$ and $\tau^S > 0$ is analogous and omitted). Then another solution of the relaxed problem is given by $(\hat{M}, \Delta^R, 0)$ with $\hat{M}_{rs} = M_{0r}$ for all $s$ and $r$.

We claim that the solution $(\hat{M}, \Delta^R, 0)$ satisfies both constraints $(S)$ and $(R)$ with equality, so that it is a solution of the original problem.

Moving from $(M, \Delta^R, \Delta^S)$ to $(\hat{M}, \Delta^R, 0)$, the quantity $\tau^S = 0$ remains unchanged, the value of the objective remains unchanged, and constraint $(S)$ becomes satisfied with equality.

Observe that at $\tau^S = 0$ constraint $(S)$ becomes

$$\Delta^S - \sum_{r \leq n-1} (M(r,0) - M(r,1)) \binom{n-1}{r} (\tau^R)^r (1-\tau^R)^{n-1-r} \quad \leq \quad 0.$$

Suppose that constraint $(R)$ is satisfied with strict inequality. Then

$$M_{r0} = 1 \quad \text{for all } r \leq n-1.$$

(If any of these equations were violated, one could slightly change the respective $M$-component, thus making the left-hand-side of $(S)$ strictly smaller than 0, followed by a slight increase of $\Delta^R$, which changes $\tau^R$ so little that both constraints remain satisfied; the increase of $\Delta^R$ increases the $g$-welfare because $g$ puts a positive weights on all types).

Hence, constraint $(R)$ becomes

$$\Delta^R - (M_{n0} - 1)(\tau^R)^{n-1} \leq 0,$$

implying $\Delta^R \leq 0$, contradicting that $\tau^R > 0$.

*Case 3.* Suppose that $\tau^S > 0$ and $\tau^R > 0$. We claim that then the solution $(M, \Delta^R, \Delta^S)$ is such that both constraints $(S)$ and $(R)$ are satisfied with equality, so that it is a solution of the original problem.

Suppose that constraint $(R)$ is satisfied with strict inequality. Then locally only constraint $(S)$ is relevant. By a similar argument as in case 2, this implies

$$M(r, n-r) = 0, \; M(r, 0) = 1, \quad \text{for all } r \leq n-1. \tag{35}$$

Moreover, if $0 < s < n - r$, then $M_{r,s} = 1$ if the following expression is $> 0$ and $M_{r,s} = 0$ if the expression is $< 0$:

$$\begin{aligned} & \binom{n-1}{r \;\; s} (\tau^R)^r (\tau^S)^s (1 - \tau^R - \tau^S)^{n-1-r-s} \\ & \quad - \binom{n-1}{r \;\; s-1} (\tau^R)^r (\tau^S)^{s-1} (1 - \tau^R - \tau^S)^{n-r-s} \\ = & \binom{n-1}{r \;\; s} (\tau^R)^r (\tau^S)^{s-1} (1 - \tau^R - \tau^S)^{n-1-r-s} \left( \tau^S - \frac{s}{n-r-s} (1 - \tau^R - \tau^S) \right). \end{aligned}$$

Because $\Delta^R > 0$ (from $\tau^R > 0$), constraint $(R)$ implies that there exists $(r, s)$ (with $s + r \leq n - 1$) such that

$$M(r+1, s) - M(r, s) \quad > \quad 0. \tag{36}$$

If $s + r = n - 1$ and $s = 0$, then (35) implies $M(r, s) = 1$, contradicting (36).

If $s + r = n - 1$ and $s > 0$, then $r + 1 \leq n - 1$ and $M(r + 1, s) = M(r + 1, n - (r + 1)) = 0$ by (35), contradicting (36).

Now consider $s+r \leq n-2$. By assumption, $M(r+1, s) > 0$ and $M(r, s) < 1$. Hence, $0 < s < n - (r + 1)$ by (35), implying

$$\tau^S - \frac{s}{n-1-r-s}(1 - \tau^R - \tau^S) \geq 0, \quad \tau^S - \frac{s}{n-r-s}(1 - \tau^R - \tau^S) \leq 0,$$

but the two inequalities are in contradiction to each other.

This completes the proof of Lemma B.

By Lemma B, any optimal voting rule $M$ is such that $(M, \Delta^R, \Delta^S)$ solves problem (relax). Let $\tau^S$ and $\tau^R$ be the participation rates corresponding to $\Delta^R$ and $\Delta^S$.

First suppose that $\tau^S > 0$ and $\tau^R > 0$.

It is not possible to change $M$ such that both constraints $(S)$ and $(R)$ become strict, because otherwise one could increase $\Delta^R$ and $\Delta^S$ slightly while keeping the constraints satisfied and increasing the objective. In other words, by the separating hyperplane theorem, there exist $\mu^R \geq 0$ and $\mu^S \geq 0$, not both equal to zero, such that $\hat{M} = M$ solves the problem

$$\max_{\hat{M}} \mu^S k^{S,\hat{M}}(\tau^R, \tau^S) + \mu^R k^{R,\hat{M}}(\tau^R, \tau^S)$$
$$\text{s.t. } 0 \leq \hat{M}_{rs} \leq 1 \quad \text{for all } (r, s).$$

The objective of this problem is linear in each $\hat{M}_{rs}$. Hence, for all $(r, s)$, if the coefficient of $\hat{M}_{rs}$ is $> 0$, then $M_{rs} = 1$, and if it is $< 0$, then $M_{rs} = 0$.

A straightforward computation now shows that

$$\text{if } k_{rs} > 0, \text{ then } M_{rs} = 1, \text{ and if } k_{rs} < 0, \text{ then } M_{rs} = 0,$$

where we use the following quantities that arise by dropping positive factors from the coefficients in the objective,

$$k_{rs} = \mu^S \tau^R((n-r-s)\tau^S - s(1-\tau^R-\tau^S)) - \mu^R \tau^S((n-r-s)\tau^R - r(1-\tau^R-\tau^S)).$$

Rearranging yields

$$k_{rs} = r\xi^R - s\xi^S - n\xi,$$

where we have used the shortcuts

$$\begin{aligned}\xi^R &= \tau^S(\mu^R - \mu^R\tau^S - \mu^S\tau^R),\\ \xi^S &= \tau^R(\mu^S - \mu^S\tau^R - \mu^R\tau^S),\\ \xi &= \tau^S\tau^R(\mu^R - \mu^S).\end{aligned}$$

Next we show that

$$\xi^R > 0, \quad \xi^S > 0.$$

In case $\mu^S \geq \mu^R$, we have

$$\xi^S \geq \mu^R \tau^R (1 - \tau^R - \tau^S) > 0.$$

From $\xi^S > 0$ it follows that $\xi^R > 0$ because otherwise $k_{rs} \leq -n\xi < 0$ for all $(r, s)$, implying $M_{rs} = 0$ for all $(r, s)$, contradicting $\tau^R > 0$ and $\tau^S > 0$. The case $\mu^S < \mu^R$ is analogous.

Now suppose that $\tau^R > 0$ and $\tau^S = 0$ (the case $\tau^R = 0$ and $\tau^S > 0$ is analogous and omitted). Defining $\check{M}(r, s) = M(r, 0)$ for all $(r, s)$, $(\check{M}, \Delta^R, \Delta^S)$ also solves problem (relax). Changing any $\check{M}_{r0}$ (while keeping $\check{M}_{rs} = \check{M}_{r0}$ for all $s$) cannot make constraint $(R)$ become strict, because otherwise one could increase $\Delta^R$ slightly while keeping the constraints satisfied and increasing the objective (using the assumption that all types have a positive $g$-welfare weight). Thus, $\hat{M}(r, 0) = M(r, 0)$ solves the problem

$$\max_{\hat{M}} k^{R,\hat{M}}(\tau^R, 0)$$
$$\text{s.t. } 0 \leq \hat{M}_{rs} \leq 1 \quad \text{for all } (r, s).$$

From this one can conclude that $M(r, 0)$ is as in an $R$-one sided default rule. This completes the proof.

*Proof of Lemma 2.* Define the equilibrium correspondence $e^M$ that assigns to each $c > 0$ the set $e^M(c)$ of pairs $(\tau^R, \tau^S)$ that are equilibria given the cost level $c$. Define $e^M(0)$ via

$$e^M(0) \;=\; \{(\tau^R, \tau^S) \mid \exists c_l \to_{l\to\infty} 0,\; c_l > 0, (\tau^R_l, \tau^S_l) \in e^M(c_l),\; (\tau^R_l, \tau^S_l) \to (\tau^R, \tau^S)\}.$$

By construction and by continuity of the equilibrium conditions, $e^M$ is an upper-hemicontinuous correspondence (cf. Berge, "Topological Spaces", p. 112, Corollary).

Consider any $(\tau^{R*}, \tau^{S*}) \in e(0)$ and a sequence $(\tau^R_l, \tau^S_l) \to (\tau^{R*}, \tau^{S*})$, $(\tau^R_l, \tau^S_l) \in e(c_l)$ for cost levels $c_l > 0$, $c_l \to 0$. Then

$$(\Delta^R_l, \Delta^S_l) = (k^R(\tau^R, \tau^S), k^S(\tau^R, \tau^S)) \to (\Delta^{R*}, \Delta^{S*}) = (k^R(\tau^{R*}, \tau^{S*}), k^S(\tau^{R*}, \tau^{S*})).$$

Consider first the case in which $M$ is an $R$-one-sided rule. Then $\tau^S_l = \tau^{S*} = 0$. Thus $\Delta^{R*} = k^R(\tau^{R*}, 0)$. If $r^* = 0$, then (iii). Suppose $r^* \geq 1$. Either $\tau^{R*} = 0$, then (vi), or $\tau^{R*} > 0$, then $\Delta^{R*} > 0$. This implies (ii) because

$$\tau^{R*} = \lim_l 1 - F(\frac{c_l}{\Delta^R_l}) = 1 - F(\frac{0}{\Delta^{R*}}) = 1 - F(0)$$

and if $M_{r^*,1} \neq 1$ then we would have $\Delta^{S*} = k^S(1 - F(0), 0) > 0$, implying $\tau^{S*} > 0$.

Similarly, if $M$ is an $S$-one-sided rule then (iv), (v), or (vi) applies.

Suppose now that $M$ is two-sided. There are four cases. First, $\tau^{R*} > 0$ and $\tau^{S*} > 0$. We will show that this implies

$$\tau^{R*} = 1 - F(0), \quad \tau^{S*} = F(0), \tag{37}$$

thus (i) holds by upper-hemicontinuity.

To see (37), suppose otherwise. Then $1 - \tau^{R*} - \tau^{S*} > 0$ so that all combinations $(r, s)$ occur with positive probability in the sum defining $k^R$. Hence, using the two-sidedness of $M$,

$$\Delta^{R*} = \lim_l k^R(\tau_l^R, \tau_l^S) = k^R(\tau^{R*}, \tau^{S*}) > 0$$

and similarly $\Delta^{S*} > 0$. Thus,

$$\tau^{R*} = \lim_l 1 - F(\frac{c_l}{\Delta_l^R}) = 1 - F(\frac{0}{\Delta^{R*}}) = 1 - F(0).$$

Similarly, $\tau^{S*} = F(0)$, implying $\tau^{R*} + \tau^{S*} = 1$, a contradiction.[20]

Second, $\tau^{R*} > 0$ and $\tau^{S*} = 0$. Thus, $e^M(c) \in [0, 1 - F(0)) \times [0, \epsilon)$ by upper-hemicontinuity. If (iii) does not hold, then $r^* \geq 1$, implying that

$$\Delta^{R*} = \lim_l k^R(\tau_l^R, \tau_l^S) = k^R(\tau^{R*}, 0) > 0.$$

Thus, as above, $\tau^{R*} = 1 - F(0)$, implying (ii) by upper-hemicontinuity.

Third, $\tau^{R*} = 0$ and $\tau^{S*} > 0$. Here, analogously to the treatment of (ii) and (iii) above, either (iv) or (v) holds.

Fourth, $\tau^{R*} = 0$ and $\tau^{S*} = 0$. Here, (vi) holds by upper hemicontinuity.

Now consider the welfare of equilibria in category (i). Assume $k \geq 2$ so that $k^{S,M}(1 - F(0), F(0)) = 0$ (the computation is similar if instead $\check{k} \geq 2$). Using

[20] Note that $(\Delta^{R*}, \Delta^{S*})$ together with $\tau^{R*} = 1 - F(0)$ and $\tau^{S*} = F(0)$ actually is an equilibrium in the setting with $c = 0$.

(10), as $c \to 0$ the welfare converges to

$$
\begin{aligned}
& E[v_i g(v_i)]\rho(A) + \int_0^{\overline{v}} v_i g(v_i) \mathrm{d}F(v_i) k^{R,M}(1-F(0),F(0)) \\
& \quad + \int_{\underline{v}}^0 (-v_i) g(v_i) \mathrm{d}F(v_i) k^{S,M}(1-F(0),F(0)) \\
= \; & E[v_i g(v_i)] \sum_{r=t^*}^{n-1} \binom{n-1}{r} (1-F(0))^r F(0)^{n-1-r} \\
& \quad + \int_0^{\overline{v}} v_i g(v_i) \mathrm{d}F(v_i) \binom{n-1}{t^*-1} (1-F(0))^{t^*-1} F(0)^{n-t^*} \\
= \; & \int_{\underline{v}}^0 v_i g(v_i) \mathrm{d}F(v_i) \sum_{r=t^*}^{n-1} \binom{n-1}{r} (1-F(0))^r F(0)^{n-1-r} \\
& \quad + \int_0^{\overline{v}} v_i g(v_i) \mathrm{d}F(v_i) \sum_{r=t^*-1}^{n-1} \binom{n-1}{r} (1-F(0))^r F(0)^{n-1-r} \\
= \; & \frac{\int_{\underline{v}}^0 v_i g(v_i) \mathrm{d}F(v_i)}{nF(0)} \sum_{r=t^*}^{n-1} \binom{n}{r} (n-r)(1-F(0))^r F(0)^{n-r} \\
& \quad + \int_0^{\overline{v}} v_i g(v_i) \mathrm{d}F(v_i) \sum_{r=t^*-1}^{n-1} \binom{n-1}{r} (1-F(0))^r F(0)^{n-1-r} \\
= \; & \frac{\int_{\underline{v}}^0 v_i g(v_i) \mathrm{d}F(v_i)}{nF(0)} \sum_{r=t^*}^{n} \binom{n}{r} (n-r)(1-F(0))^r F(0)^{n-r} \\
& \quad + \frac{\int_0^{\overline{v}} v_i g(v_i) \mathrm{d}F(v_i)}{n(1-F(0))} \sum_{r=t^*-1}^{n-1} \binom{n}{r+1} (r+1)(1-F(0))^{r+1} F(0)^{n-(r+1)} \\
= \; & \frac{1}{n} \sum_{r=t^*}^{n} \binom{n}{r} (1-F(0))^r F(0)^{n-r} \left(-(n-r)E_i + rE_+\right) \\
= \; & w_g(t^*),
\end{aligned}
$$

as was to be shown. The proofs of the other welfare statements are similar. This completes the proof.

*Proof of Lemma 3.* We have to deal with up to $k$th order partial derivatives of $k^R$, $k^S$, and $\rho(A)$ with respect to $\tau^R$ and $\tau^S$. We begin with these. We will use the lower index $(l)\tau^S$ ($l \geq 1$) for the $l$th partial derivative with respect to $\tau^S$, evaluated at $(\tau^R, \tau^S) = (1 - F(0), F(0))$. Similar for partial derivatives with respect to $\tau^R$. We note here expressions for these derivatives that are computed below. We will use the shortcut

$$x = \frac{(n-1)!}{(t^*-1)!\hat{k}!}(1-F(0))^{t^*-1}F(0)^{\hat{k}}(-1)^{k-1}.$$

For all $l \geq 0$,[21]

$$k^S_{(l)\tau^S} = x \cdot \begin{cases} 0 & \text{if } 1 \leq l \leq k-2, \\ 1 & \text{if } l = k-1, \\ -\mathbf{1}_{M_{t^*-2,\hat{k}}=1} \cdot \frac{t^*-1}{1-F(0)} + \frac{k}{F(0)} & \text{if } l = k. \end{cases} \tag{38}$$

Moreover,

$$\begin{aligned} k^S_{(1)\tau^R} &= -\mathbf{1}_{k=2} \left( \begin{array}{c} n-1 \\ t^*-1 \;\; \hat{k} \end{array} \right) (1-F(0))^{t^*-1}F(0)^{\hat{k}} \\ &= -\mathbf{1}_{k=2} \cdot \Delta^{t*}\frac{n-t^*}{F(0)}. \end{aligned} \tag{39}$$

For all $l \geq 0$,

$$k^R_{(l)\tau^S} = x \cdot \begin{cases} 0 & \text{if } 1 \leq l \leq k-2, \\ -1, & \text{if } l = k-1, \\ -\mathbf{1}_{M_{t^*-2,\hat{k}}=0}\frac{t^*-1}{1-F(0)} - \frac{\hat{k}}{F(0)}(k-1) & \text{if } l = k, \end{cases} \tag{40}$$

Moreover,

$$k^R_{(1)\tau^R} = \Delta^{t*}\frac{t^*-1}{1-F(0)} - \mathbf{1}_{k>2} \cdot \Delta^{t*}\frac{n-t^*}{F(0)}. \tag{41}$$

For all $l \geq 0$,

$$\rho(A)_{(l)\tau^S} = x \cdot \begin{cases} 0, & \text{if } 1 \leq l \leq k-2, \\ 1, & \text{if } l = k-1, \\ -\mathbf{1}_{M_{t^*-2,\hat{k}}=1} \cdot \frac{t^*-1}{1-F(0)} + \frac{\hat{k}(k-1)}{F(0)} & \text{if } l = k. \end{cases} \tag{42}$$

Finally,

$$\rho(A)_{(1)\tau^R} = \mathbf{1}_{k>2} \cdot \Delta^{t*}\frac{n-t^*}{F(0)}. \tag{43}$$

[21]Note that, if $k > 2$ then $M_{t^*-2,\hat{k}} = 0$ by linearity.

*Proof of (38).* Towards computing derivatives at $c = 0$, the size of the exponent of $(1 - \tau^R - \tau^S)$ plays a crucial role. At $c = 0$, for any $l \geq 0$, the only non-vanishing derivative of $(1 - \tau^R - \tau^S)^l$ is the $l$th derivative.

$$\begin{aligned} k^S &= \left( \sum_{r+s \leq n-k-2} \dots \right) \\ &\quad + \mathbf{1}_{t^*>1,\ M_{t^*-2,\hat{k}}=1} \binom{n-1}{t^*-2\ \ \hat{k}} (\tau^R)^{t^*-2} (\tau^S)^{\hat{k}} (1 - \tau^R - \tau^S)^k \\ &\quad + \binom{n-1}{t^*-1\ \ \hat{k}} (\tau^R)^{t^*-1} (\tau^S)^{\hat{k}} (1 - \tau^R - \tau^S)^{k-1}. \end{aligned} \tag{44}$$

All terms in $(\sum \dots)$ vanish if we take the $l$th derivative ($l \leq k$) and evaluate at $c = 0$ (i.e., at $\tau^R + \tau^S = 1$) because $(1 - \tau^R - \tau^S)$ occurs in all terms with an exponent $> k$.

Applying the general Leibniz product rule to the second and third row in (44), we obtain

$$k^S_{(l)\tau^S} = \begin{cases} 0 & \text{if } 1 \leq l \leq k-2, \\ \binom{n-1}{t^*-1\ \ \hat{k}} (1 - F(0))^{t^*-1} F(0)^{\hat{k}} (-1)^{k-1} (k-1)! & \text{if } l = k-1, \\ \mathbf{1}_{t^*>1,\ M_{t^*-2,\hat{k}}=1} \binom{n-1}{t^*-2\ \ \hat{k}} (1 - F(0))^{t^*-2} F(0)^{\hat{k}} k! (-1)^k & \\ +\mathbf{1}_{\hat{k}>0} \binom{n-1}{t^*-1\ \ \hat{k}} (1 - F(0))^{t^*-1} \hat{k} F(0)^{\hat{k}-1} (-1)^{k-1} (k-1)! k & \text{if } l = k, \end{cases}$$

From this one obtains (38).

*Proof of (40).* Note that

$$\begin{aligned} k^R &= \left( \sum_{r+s \leq n-k-2} \dots \right) \\ &\quad + \mathbf{1}_{t^*>1,\ M_{t^*-2,\hat{k}}=0} \binom{n-1}{t^*-2\ \ \hat{k}} (\tau^R)^{t^*-2} (\tau^S)^{\hat{k}} (1 - \tau^R - \tau^S)^k \\ &\quad + \sum_{\hat{s}=0}^{k-2} \binom{n-1}{t^*-1\ \hat{s}} (\tau^R)^{t^*-1} (\tau^S)^{n-t^*-\hat{s}} (1 - \tau^R - \tau^S)^{\hat{s}}, \end{aligned} \tag{45}$$

where we have used the parameter $\hat{s} = n - t^* - s$ instead of $s$ to write the last sum.[22]

Taking the $l$th ($l \leq k$) derivative of (45) and evaluating at $1 - \tau^R - \tau^S = 0$, all terms in the first row vanish because $(1 - \tau^R - \tau^S)$ occurs in all terms with an exponent $> k$. Similarly, the second row vanishes unless $l = k$.

We begin by showing (40) for $1 \leq l \leq k - 2$. Consider the $l$th derivative of the third row in (45), evaluated at $1 - \tau^R - \tau^S = 0$. Within the $l$th derivative expression as represented according to the general Leibniz product rule, only the term resulting from taking the $\hat{s}$th derivative of $(1 - \tau^R - \tau^S)^{\hat{s}}$ (and taking the $(l - \hat{s})$th derivative of $(\tau^S)^{n-t^*-\hat{s}}$) does not vanish. Thus,

$$
\begin{aligned}
& k^R_{(l)\tau^S} \\
&= \sum_{\hat{s}=0}^{l} \left( \begin{array}{c} n-1 \\ t^*-1 \;\; \hat{s} \end{array} \right) (1-F(0))^{t^*-1} \frac{(n-t^*-\hat{s})!}{(n-t^*-\hat{s}-(l-\hat{s}))!} F(0)^{n-t^*-\hat{s}-(l-\hat{s})} \\
&\qquad \cdot \hat{s}!(-1)^{\hat{s}} \left( \begin{array}{c} l \\ \hat{s} \end{array} \right) \\
&= (1-F(0))^{t^*-1} F(0)^{n-t^*-l} \sum_{\check{s}=0}^{l} \frac{(n-1)!}{(t^*-1)!(n-t^*-\check{s})!\check{s}!} \frac{(n-t^*-\check{s})!}{(n-t^*-l)!} \\
&\qquad \cdot (\check{s})!(-1)^{\check{s}} \left( \begin{array}{c} l \\ \check{s} \end{array} \right) \\
&= (1-F(0))^{t^*-1} F(0)^{n-t^*-l} \sum_{\check{s}=0}^{l} \frac{(n-1)!}{(t^*-1)!} \frac{1}{(n-t^*-l)!} \cdot (-1)^{\check{s}} \left( \begin{array}{c} l \\ \check{s} \end{array} \right) \\
&= (1-F(0))^{t^*-1} F(0)^{n-t^*-l} \frac{(n-1)!}{(t^*-1)!} \frac{1}{(n-t^*-l)!} \sum_{\check{s}=0}^{l} (-1)^{\check{s}} \left( \begin{array}{c} l \\ \check{s} \end{array} \right) \\
&= 0.
\end{aligned} \tag{46}
$$

[22]Observe that $k - 1 \leq n - t^*$ by construction of $k$.

To show (40) for $l = k-1$, we use (45) and the general Leibniz product rule,

$$\begin{aligned}
k^R_{(k-1)\tau^S} &= \sum_{\hat{s}=0}^{k-2} \begin{pmatrix} n-1 \\ t^*-1\ \hat{s} \end{pmatrix} (1-F(0))^{t^*-1} F(0)^{\overbrace{n-t^*-\hat{s}-(k-1-\hat{s})}^{=n-t^*-(k-1)=\hat{k}}} \\
&\quad \cdot \frac{(n-t^*-\hat{s})!}{(n-t^*-(k-1))!} \hat{s}!(-1)^{\hat{s}} \begin{pmatrix} k-1 \\ \hat{s} \end{pmatrix} \\
&= \begin{pmatrix} n-1 \\ t^*-1\ \hat{k} \end{pmatrix} (k-1)!(1-F(0))^{t^*-1} F(0)^{\hat{k}} \cdot \sum_{\hat{s}=0}^{k-2} (-1)^{\hat{s}} \begin{pmatrix} k-1 \\ \hat{s} \end{pmatrix} \\
&= -x,
\end{aligned}$$

where we have used the identity $\sum_{\hat{s}=0}^{k-1} (-1)^{\hat{s}} \begin{pmatrix} k-1 \\ \hat{s} \end{pmatrix} = 0$.

To show (40) for $l = k$, note that

$$\begin{aligned}
k^R_{(k)\tau^S} &= \mathbf{1}_{t^*>1,\ M_{t^*-2,\hat{k}}=0} \begin{pmatrix} n-1 \\ t^*-2\ \ \hat{k} \end{pmatrix} (1-F(0))^{t^*-2} F(0)^{\hat{k}} k!(-1)^k \\
&\quad + \mathbf{1}_{\hat{k}>0} \sum_{\hat{s}=0}^{k-2} \begin{pmatrix} n-1 \\ t^*-1\ \hat{s} \end{pmatrix} (1-F(0))^{t^*-1} F(0)^{\overbrace{n-t^*-\hat{s}-(k-\hat{s})}^{=n-t^*-k=\hat{k}-1}} \\
&\quad \cdot \frac{(n-t^*-\hat{s})!}{(n-t^*-k)!} \hat{s}!(-1)^{\hat{s}} \begin{pmatrix} k \\ \hat{s} \end{pmatrix} \\
&= -\mathbf{1}_{t^*>1,\ M_{t^*-2,\hat{k}}=0} \cdot x \frac{t^*-1}{1-F(0)} \\
&\quad + \mathbf{1}_{\hat{k}>0} \begin{pmatrix} n-1 \\ t^*-1\ \hat{k}-1 \end{pmatrix} k!(1-F(0))^{t^*-1} F(0)^{\hat{k}-1} \cdot \sum_{\hat{s}=0}^{k-2} (-1)^{\hat{s}} \begin{pmatrix} k \\ \hat{s} \end{pmatrix} \\
&= -\mathbf{1}_{t^*>1,\ M_{t^*-2,\hat{k}}=0} \cdot x \frac{t^*-1}{1-F(0)} - \mathbf{1}_{\hat{k}>0} \cdot x \frac{\hat{k}}{F(0)} (k-1),
\end{aligned}$$

where we have used the identity

$$\sum_{\hat{s}=0}^{k-2} (-1)^{\hat{s}} \begin{pmatrix} k \\ \hat{s} \end{pmatrix} = -\sum_{\hat{s}=k-1}^{k} (-1)^{\hat{s}} \begin{pmatrix} k \\ \hat{s} \end{pmatrix} = -(-1)^{k-1} k - (-1)^k = (-1)^k (k-1).$$

This completes the proof of (40).

*Proof of (47).* Using (45),

$$\begin{aligned} k^R_{(1)\tau^R} &= \mathbf{1}_{t^*>1} \binom{n-1}{t^*-1\;0} (t^*-1)(1-F(0))^{t^*-2} F(0)^{n-t^*} \\ &\quad - \mathbf{1}_{k>2} \binom{n-1}{t^*-1\;1} (1-F(0))^{t^*-1} F(0)^{n-t^*-1} \\ &= \Delta^{t*} \frac{t^*-1}{1-F(0)} - \mathbf{1}_{k>2} \cdot \Delta^{t*} \frac{n-t^*}{F(0)}. \end{aligned} \tag{47}$$

*Proof of (42).* Note that

$$\begin{aligned} \rho(A) &= \sum_{r+s\leq n-1} \binom{n-1}{r\;s} (\tau^R)^r (\tau^S)^s (1-\tau^R-\tau^S)^{n-1-r-s} M_{rs} \\ &= \left( \sum_{r+s\leq n-2-k} \cdots \right) \\ &\quad + \mathbf{1}_{\hat{k}>0} \binom{n-1}{t^*-1\;\hat{k}-1} (\tau^R)^{t^*-1} (\tau^S)^{\hat{k}-1} (1-\tau^R-\tau^S)^k \\ &\quad + \mathbf{1}_{t^*>1,\; M_{t^*-2,\hat{k}}=1} \binom{n-1}{t^*-2\;\hat{k}} (\tau^R)^{t^*-2} (\tau^S)^{\hat{k}} (1-\tau^R-\tau^S)^k \\ &\quad + \binom{n-1}{t^*-1\;\hat{k}} (\tau^R)^{t^*-1} (\tau^S)^{\hat{k}} (1-\tau^R-\tau^S)^{k-1} \\ &\quad + \sum_{r+s\geq n-1-k,\; r\geq t^*} \binom{n-1}{r\;s} (\tau^R)^r (\tau^S)^s (1-\tau^R-\tau^S)^{\overbrace{n-1-r-s}^{\leq k}} \end{aligned} \tag{48}$$

Taking the $l$th derivative ($1 \leq l \leq k-2$), only terms in the last sum can be non-vanishing because $l < k-1$. In the last sum, any term with $n-1-r-s > l$ vanishes, and any term with $s + (n-1-r-s) < l$ vanishes. Thus, using the general Leibniz product rule,

$$
\begin{aligned}
\rho(A)_{(l)\tau^S} &= \sum_{n-1-l\le r+s\le n-1,\ r\ge t^*,\ n-1-r\ge l} \begin{pmatrix} n-1 \\ r\ s \end{pmatrix} (1-F(0))^r \begin{pmatrix} l \\ n-1-r-s \end{pmatrix} \\
&\quad \cdot \frac{s!}{(n-1-r-l)!} F(0)^{\overbrace{s-(l-(n-1-r-s))}^{=n-1-r-l}} (n-1-r-s)!(-1)^{n-1-r-s} \\
&= \sum_{r=t^*}^{n-1-l} (1-F(0))^r \frac{(n-1)!}{r!(n-1-r-l)!} F(0)^{n-1-r-l} \\
&\qquad \cdot \sum_{n-1-l-r\le s\le n-1-r} \begin{pmatrix} l \\ n-1-r-s \end{pmatrix} (-1)^{n-1-r-s}.
\end{aligned}
$$

The last sum equals 0, as can be seen by using the variable $\check{s} = n-1-r-s$ instead of $s$. This shows (42) for $1 \le l \le k-2$.

The above computation also works if $l = k-1$ or $l = k$, showing that the fifth row on the right-hand-side of (48) can be ignored.

Consider $l = k-1$. The $(k-1)$th derivative of the fourth row on the right-hand-side of (48) equals $x$, while the $(k-1)$th derivatives of the second and third rows vanish.

Consider $l = k$. The $k$th derivative of the second and third rows on the right-hand-side of (48) are obtained by taking the $k$th derivative of $(1-\tau^R-\tau^S)^k$, yielding the terms

$$
\begin{aligned}
& \mathbf{1}_{\hat{k}>0} \begin{pmatrix} n-1 \\ t^*-1\ \hat{k}-1 \end{pmatrix} (1-F(0))^{t^*-1} F(0)^{\hat{k}-1} k!(-1)^k \\
&= \mathbf{1}_{\hat{k}>0} \frac{(n-1)!}{(t^*-1)!(\hat{k}-1)!} (1-F(0))^{t^*-1} F(0)^{\hat{k}-1} (-1)^k \\
&= -\mathbf{1}_{\hat{k}>0} \cdot x \frac{\hat{k}}{F(0)}.
\end{aligned} \tag{49}
$$

and

$$
\begin{aligned}
& \mathbf{1}_{t^*>1,\ M_{t^*-2,\hat{k}}=1} \begin{pmatrix} n-1 \\ t^*-2\ \hat{k} \end{pmatrix} (1-F(0))^{t^*-2} F(0)^{\hat{k}} k!(-1)^k \\
&= \mathbf{1}_{t^*>1,\ M_{t^*-2,\hat{k}}=1} \cdot \frac{(n-1)!}{(t^*-2)!\hat{k}!} (1-F(0))^{t^*-2} F(0)^{\hat{k}} (-1)^k \\
&= -\mathbf{1}_{t^*>1,\ M_{t^*-2,\hat{k}}=1} \cdot x \frac{t^*-1}{1-F(0)}.
\end{aligned}
$$

The last remaining term is obtained by taking the $k$th derivative of the fourth row on the right-hand-side of (48). Using the Leibniz product rule, we take the $(k-1)$th derivative of $(1-\tau^R-\tau^S)^{k-1}$ and the 1st derivative of $(\tau^S)^{\hat{k}}$ and multiply with $\left(\begin{array}{c} k \\ 1 \end{array}\right) = k$, yielding the term

$$\begin{aligned}
& \mathbf{1}_{\hat{k}>0} \left(\begin{array}{c} n-1 \\ t^*-1\ \hat{k} \end{array}\right) (1-F(0))^{t^*-1} \hat{k} F(0)^{\hat{k}-1} (k-1)! (-1)^{k-1} k \\
= \; & \mathbf{1}_{\hat{k}>0} \frac{(n-1)!}{(t^*-1)!(\hat{k}-1)!} (1-F(0))^{t^*-1} F(0)^{\hat{k}-1} (-1)^{k-1} k \\
= \; & \mathbf{1}_{\hat{k}>0} \cdot x \frac{\hat{k}}{F(0)} k.
\end{aligned}$$

Summarizing this with (49), we obtain the last term in (42). This completes the proof of (42).

*Proof of (43).*

$$\begin{aligned}
\rho(A)_{(1)\tau^R} \; = \; & \mathbf{1}_{k=2} \left(\begin{array}{c} n-1 \\ t^*-1\ \hat{k} \end{array}\right) (1-F(0))^{t^*-1} F(0)^{\hat{k}} (-1) \\
& + \sum_{r+s=n-2,\ r \geq t^*} \left(\begin{array}{c} n-1 \\ r\ s \end{array}\right) (1-F(0))^r F(0)^s (-1) \\
& + \underbrace{\sum_{r+s=n-1,\ r \geq t^*,\ r \geq 1} \left(\begin{array}{c} n-1 \\ r\ s \end{array}\right) r (1-F(0))^{r-1} F(0)^s}_{= \sum_{\hat{r}+s=n-2,\ \hat{r} \geq t^*-1} \left(\begin{array}{c} n-1 \\ \hat{r}\ s \end{array}\right) (1-F(0))^{\hat{r}} F(0)^s, \text{ where } \hat{r} = r-1} \\
= \; & -\mathbf{1}_{k=2} \left(\begin{array}{c} n-1 \\ t^*-1\ \hat{k} \end{array}\right) (1-F(0))^{t^*-1} F(0)^{\hat{k}} \\
& + \left(\begin{array}{c} n-1 \\ t^*-1\ n-1-t^* \end{array}\right) (1-F(0))^{t^*-1} F(0)^{n-1-t^*} \\
= \; & -\mathbf{1}_{k=2} \cdot \Delta^{t*} \frac{n-t^*}{F(0)} + \Delta^{t*} \frac{n-t^*}{F(0)} \\
= \; & \mathbf{1}_{k>2} \cdot \Delta^{t*} \frac{n-t^*}{F(0)}.
\end{aligned}$$

We are now prepared to complete the proof of Lemma 3. Using the shortcut $\xi = (1 - F(0), F(0))$ and the definition of $k^R$,

$$\begin{aligned} k^R(\xi) &= \sum_{r+s=n-1} (M(r+1,s) - M(r,s)) \left( \begin{array}{c} n-1 \\ r \;\; s \end{array} \right) (1-F(0))^r F(0)^s \\ &= \Delta^{t*} > 0. \end{aligned} \tag{50}$$

Similarly, using $k \geq 2$,

$$k^S(\xi) = 0 \stackrel{\text{def}}{=} \Delta^{S*}. \tag{51}$$

Consider in a neighborhood of the point $(0, \xi)$ the function

$$\phi(c, \tau^R, \tau^S) = \left( \begin{array}{c} F^{-1}(1-\tau^R) k^R(\tau^R, \tau^S) - c \\ F^{-1}(\tau^S) k^S(\tau^R, \tau^S) + c \end{array} \right).$$

Any solution to the system of equations

$$\phi(c, \tau^R, \tau^S) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \tag{52}$$

with $c > 0$, $0 < \tau^R < 1 - F(0)$ and $0 < \tau^S < F(0)$ yields an equilibrium $(\tau^R, \tau^S)$ at participation cost $c$. To see this, define $\Delta^R = k^R(\tau^R, \tau^S)$ and $\Delta^S = k^S(\tau^R, \tau^S)$, implying $\tau^R = l^R(\Delta^R)$ and $\tau^S = l^S(\Delta^S)$.

Vice versa, for any $(\tau^R, \tau^S) \in e^M(c)$ at some $c > 0$ with $\tau^R > 0$ and $\tau^S > 0$, the equations (52) are satisfied.

The point $(0, \xi)$ solves (52).

Towards completing the proof, the crucial step is that (52) has a locally unique solution for all sufficiently small $c > 0$. To show this, we will apply the Implicit Functions Theorem (IFT).

The derivative of $\phi$ with respect to the first two variables is

$$\phi_{\partial c, \partial \tau^R} = \left( \begin{array}{ll} -1 & -(F^{-1})'(1-\tau^R) k^R + F^{-1}(1-\tau^R) k^R_{\tau^R} \\ 1 & F^{-1}(\tau^S) k^S_{\tau^R}, \end{array} \right) \tag{53}$$

where lower indices denote partial derivatives.

The determinant of the matrix (53) is

$$D(\tau^R, \tau^S) = (F^{-1})'(1-\tau^R) k^R - F^{-1}(1-\tau^R) k^R_{\tau^R} - F^{-1}(\tau^S) k^S_{\tau^R}.$$

In particular,

$$\phi_{\partial c,\partial\tau^R}(0,\xi) \quad = \quad \begin{pmatrix} -1 & -\frac{1}{f(0)}\Delta^{t*} \\ 1 & 0 \end{pmatrix} \tag{54}$$

and

$$D(\xi) \quad = \quad \frac{\Delta^{t*}}{f(0)}.$$

Because $D(\xi) \neq 0$ by (50), the IFT implies that, for some $\epsilon^S > 0$, there exist functions $c(\tau^S)$ and $\tau^R(\tau^S)$ ($\tau^S \in (F(0) - \epsilon^S, F(0)]$) such that (52) is solved.[23] [24] Moreover, the IFT states the following. There exists $c' > 0$ and $\epsilon^R > 0$ such that $c(\tau^S) \in (-c', c')$ and $\tau^R \in (1 - F(0) - \epsilon^R, 1 - F(0) + \epsilon^R)$ for all $\tau^S \in (F(0) - \epsilon^S, F(0)]$, and the following uniqueness statement holds:

> if $c \in (-c', c')$, $\tau^R \in (1 - F(0) - \epsilon^R, 1 - F(0) + \epsilon^R]$, $\tau^S \in (F(0) - \epsilon^S, F(0)]$,
> and (52),
> then $c = c(\tau^S)$ and $\tau^R = \tau^R(\tau^S)$.

Without loss of generality, $\epsilon^S \leq \epsilon^R$.

In order to actually compute derivatives, we need the inverted matrix

$$\phi^{-1}_{\partial c,\partial\tau^R} \quad = \quad \frac{1}{D(\tau^R,\tau^S)} \begin{pmatrix} F^{-1}(\tau^S)k^S_{\tau^R} & (F^{-1})'(1-\tau^R)k^R - F^{-1}(1-\tau^R)k^R_{\tau^R} \\ -1 & -1 \end{pmatrix}$$

and the derivative

$$\phi_{\partial\tau^S} \quad = \quad \begin{pmatrix} F^{-1}(1-\tau^R)k^R_{\tau^S} \\ F^{-1}(\tau^S)k^S_{\tau^S} + (F^{-1})'(\tau^S)k^S \end{pmatrix}.$$

From the IFT, in a neighborhood of $\tau^S = F(0)$,

$$\begin{pmatrix} \mathrm{d}c/\mathrm{d}\tau^S \\ \mathrm{d}\tau^R/\mathrm{d}\tau^S \end{pmatrix} \quad = \quad -\phi^{-1}_{\partial c,\partial\tau^R} \circ \phi_{\partial\tau^S}. \tag{55}$$

At $\xi$, we have $\phi_{(1)\tau^S} = (0,0)^T$. Thus (55) implies (14) for $l = 1$.

---

[23] Using (51), the reader may verify that the IFT cannot be applied if we take $c$ (or $\tau^R$) as the independent variable because the relevant determinant equals 0.

[24] The IFT also implies that the functions extend to the right of $\tau^S = F(0)$, which is irrelevant.

We proceed by induction over $l$. Suppose the formula in (14) holds for some $l$ and we want to show it for $l+1$, where $l+1 < k$. Applying the chain rule and general Leibniz product rule to (55), it is sufficient to show $\phi_{(l')_{\tau^S}} = (0,0)^T$ for all $l' \leq l+1$. Consider the first factor, $F^{-1}(1-\tau^R)$, of the first component of $\phi_{\partial_{\tau^S}}$. By the chain rule and the induction hypothesis, the first $l$ derivatives of this factor vanish at $\tau^S = F(0)$. Hence, the first $l$ derivatives of the first component of $\phi_{\partial_{\tau^S}}$ vanish. Of the second component of $\phi_{\partial_{\tau^S}}$, the term $F^{-1}(\tau^S)$ vanishes at $\tau^S = F(0)$, and, because $l < k-1$, by (38), the first $l$ derivatives of $k^S$ (with respect to $\tau^S$) also vanish at $\tau^S = F(0)$. Hence, the first $l$ derivatives of the second component of $\phi_{\partial_{\tau^S}}$ vanish. This completes the induction.

From (55),

$$\begin{pmatrix} \mathrm{d}^k c/\mathrm{d}(\tau^S)^k \\ \mathrm{d}^k \tau^R/\mathrm{d}(\tau^S)^k \end{pmatrix} = -\frac{\mathrm{d}^{k-1}}{\mathrm{d}(\tau^S)^{k-1}} \left(\phi^{-1}_{\partial_c,\partial_{\tau^R}} \circ \phi_{\partial_{\tau^S}}\right).$$

Because $\phi_{(l')_{\tau^S}} = (0,0)^T$ for all $l' \leq k-1$ from the induction above,

$$\begin{aligned} \begin{pmatrix} \left.\frac{\mathrm{d}^k c}{\mathrm{d}(\tau^S)^k}\right|_{\tau^S=F(0)} \\ \left.\frac{\mathrm{d}^k \tau^R}{\mathrm{d}(\tau^S)^k}\right|_{\tau^S=F(0)} \end{pmatrix} &= -\left.\phi^{-1}_{\partial_c,\partial_{\tau^R}}\right|_{\xi} \circ \phi_{(k)_{\tau^S}} \\ &= -\frac{f(0)}{\Delta^{t*}} \begin{pmatrix} 0 & \frac{\Delta^{t*}}{f(0)} \\ -1 & -1 \end{pmatrix} \circ \frac{\mathrm{d}^{k-1}}{\mathrm{d}(\tau^S)^{k-1}} \left.\begin{pmatrix} F^{-1}(1-\tau^R)k^R_{\tau^S} \\ F^{-1}(\tau^S)k^S_{\tau^S} + (F^{-1})'(\tau^S)k^S. \end{pmatrix}\right|_{\tau^S=F(0)} \end{aligned} \tag{56}$$

By (14) and the chain rule,

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}(\tau^S)^{k-1}} \left.\left(F^{-1}(1-\tau^R)k^R_{\tau^S}\right)\right|_{\tau^S=F(0)} = 0$$

and, using the general Leibniz product rule,

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}(\tau^S)^{k-1}} \left.\left(F^{-1}(\tau^S)k^S_{\tau^S}\right)\right|_{\tau^S=F(0)} = (k-1)\frac{1}{f(0)}k^S_{(k-1)_{\tau^S}} \overset{(38)}{=} \frac{k-1}{f(0)}x.$$

Also,

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}(\tau^S)^{k-1}} \left.\left((F^{-1})'(\tau^S)k^S\right)\right|_{\tau^S=F(0)} = \frac{1}{f(0)}x.$$

Thus, (56) implies

$$\left(\begin{array}{c} \frac{\mathrm{d}^k c}{\mathrm{d}(\tau^S)^k}\Big|_{\tau^S=F(0)} \\ \frac{\mathrm{d}^k \tau^R}{\mathrm{d}(\tau^S)^k}\Big|_{\tau^S=F(0)} \end{array}\right) = -\frac{f(0)}{\Delta^{t*}}\left(\begin{array}{cc} 0 & \frac{\Delta^{t*}}{f(0)} \\ -1 & -1 \end{array}\right) \circ \left(\begin{array}{c} 0 \\ \frac{k}{f(0)}x \end{array}\right),$$

yielding (73).

From (14), (73), and (38),

$$(-1)^l \frac{\mathrm{d}^l c}{\mathrm{d}(\tau^S)^l}\Big|_{\tau^S=F(0)} \quad \begin{cases} = 0 & \text{if } l < k, \\ > 0 & \text{if } l = k. \end{cases} \tag{57}$$

Thus, there exists $0 < \overline{\epsilon} \leq \epsilon^S$ such that $c(\tau^S)$ is strictly decreasing for $\tau^S \in (F(0) - \overline{\epsilon}, F(0)]$. (Showing that this is enough is a simple calculus lemma. If a $k$ times continuously differentiable function $h(x)$ has $k-1$ derivatives equal to 0 at $x = 0$ and the $k$th derivative is strictly positive at 0, then the function is strictly increasing in a right-neighborhood of 0. Denoting derivatives with a lower index, $h_{k-1}(x) = \int_0^x h_k(y)\mathrm{d}y > 0$ by the fundamental theorem of calculus, hence $h_{k-2}(x) = \int_0^x h_{k-1}(y)\mathrm{d}y > 0$, and so on continuing inductively until we find $h_1(x) > 0$ for all $x$ in a right neighborhood of 0, showing that $h$ is strictly increasing. Now consider the function $\hat{h}(x) = h(-x)$ in a left-neighborhood of 0, then the $k$th derivative $\hat{h}_k(0) = (-1)^k h_k(0)$. So, if $(-1)^k \hat{h}_k(0) > 0$ then $\hat{h}$ is strictly decreasing in a left neighborhood of 0.)

Similarly, we can assume that $\tau^R(\tau^S)$ is strictly increasing for $\tau^S \in (F(0) - \overline{\epsilon}, F(0)]$.

Let $\overline{c} = c(F(0) - \overline{\epsilon}) > 0$. By construction, $\overline{c} \leq c'$.

Because $c(\tau^S)$ is strictly decreasing and continuous, it has a strictly decreasing and continuous inverse $\tau^S(c)$ ($c \in [0, \overline{c})$). Defining $\tau^R(c) = \tau^R(\tau^S(c))$ ($c \in [0, \overline{c})$), it follows that $\tau^R(c)$ is strictly decreasing and continuous.

The differentiability statements follow by construction.

The second limit in (13) is immediate from (14) with $l = 1$. To obtain the first limit, consider the function $\tau^R(c) = \tau^R(\tau^S(c))$ (abusing notation). Then

$$(\tau^R)'(c) = (\tau^R)'(\tau^S(c))(\tau^S)'(c) = \frac{(\tau^R)'(\tau^S(c))}{c'(\tau^S(c))}.$$

Thus,

$$\lim_{c\to 0}(\tau^R)'(c) = \lim_{\tau^S \to F(0)} = \frac{(\tau^R)'(\tau^S)}{c'(\tau^S)}.$$

Applying L'Hospital's rule $k$ times and using (14) and (15),

$$\lim_{c \to 0} (\tau^R)'(c) = \frac{\frac{\mathrm{d}^k \tau^R}{\mathrm{d}(\tau^S)^k}\Big|_{\tau^S = F(0)}}{\frac{\mathrm{d}^k c}{\mathrm{d}(\tau^S)^k}\Big|_{\tau^S = F(0)}} = -\frac{f(0)}{\Delta^{t*}}.$$

Lastly, we prove (17). We begin by writing the welfare as a function of $\tau^S$,

$$\begin{aligned}\hat{W}(\tau^S) &= \rho(A)E[vg(v)] + \int_{F^{-1}(1-\tau^R)}^{\overline{v}} (\Delta^R(\tau^S)v - c(\tau^S))g(v)\mathrm{d}F(v) \\ &\quad + \int_{\underline{v}}^{F^{-1}(\tau^S)} (\Delta^S(\tau^S)(-v) - c(\tau^S))g(v)\mathrm{d}F(v),\end{aligned}$$

where $\tau^R$ and $c$ are functions of $\tau^S$, and

$$\Delta^R(\tau^S) = \frac{c}{F^{-1}(1-\tau^R(\tau^S))}, \quad \Delta^S(\tau^S) = \frac{-c}{F^{-1}(\tau^S)}. \tag{58}$$

Using the equations (52), there is another way of representing these functions,

$$\Delta^R(\tau^S) = k^R(\tau^R(\tau^S), \tau^S), \quad \Delta^S(\tau^S) = k^S(\tau^R(\tau^S), \tau^S).$$

Towards taking the derivative of $\hat{W}$, observe that from (58) the integrand equals 0 at the variable boundary of the respective integration area. Thus,

$$\begin{aligned}\frac{\mathrm{d}\hat{W}}{\mathrm{d}\tau^S} &= \frac{\mathrm{d}\rho(A)}{\mathrm{d}\tau^S}E[vg(v)] \\ &\quad + \int_{F^{-1}(1-\tau^R)}^{\overline{v}} ((\Delta^R)'(\tau^S)v - c'(\tau^S))g(v)\mathrm{d}F(v) \\ &\qquad + \int_{\underline{v}}^{F^{-1}(\tau^S)} ((\Delta^S)'(\tau^S)(-v) - c'(\tau^S))g(v)\mathrm{d}F(v). \\ &= \frac{\mathrm{d}\rho(A)}{\mathrm{d}\tau^S}E[v] \\ &\quad + (\Delta^R)'(\tau^S)\int_{F^{-1}(1-\tau^R)}^{\overline{v}} vg(v)\mathrm{d}F(v) + (\Delta^S)'(\tau^S)\int_{\underline{v}}^{F^{-1}(\tau^S)} (-v)g(v)\mathrm{d}F(v). \\ &\quad - \int_{F^{-1}(1-\tau^R)}^{\overline{v}} g(v)\mathrm{d}F(v)c'(\tau^S) - \int_{\underline{v}}^{F^{-1}(\tau^S)} g(v)\mathrm{d}F(v)c'(\tau^S).\end{aligned} \tag{59}$$

The difficult terms in the derivative are

$$\frac{\mathrm{d}\rho(A)}{\mathrm{d}\tau^S} = \frac{\partial\rho(A)}{\partial\tau^R}\cdot(\tau^R)'(\tau^S) + \frac{\partial\rho(A)}{\partial\tau^S}, \tag{60}$$

$$(\Delta^R)'(\tau^S) = \frac{\partial k^R}{\partial\tau^R}\cdot(\tau^R)'(\tau^S) + \frac{\partial k^R}{\partial\tau^S}, \tag{61}$$

$$(\Delta^S)'(\tau^S) = \frac{\partial k^S}{\partial\tau^R}\cdot(\tau^R)'(\tau^S) + \frac{\partial k^S}{\partial\tau^S}. \tag{62}$$

Using (60), (42) and (14),

$$\left.\frac{\mathrm{d}^l\rho(A)}{\mathrm{d}(\tau^S)^l}\right|_{\tau^S=F(0)} = 0 \quad \text{for all } 1 \le l \le k-2. \tag{63}$$

Using (61), (40) and (14), and denoting the $l$th derivatives evaluated at $F(0)$ by upper indices,

$$(\Delta^R)^{(l)} = 0 \quad \text{for all } 1 \le l \le k-2. \tag{64}$$

Similarly, (62), (38), and (14),

$$(\Delta^S)^{(l)} = 0 \quad \text{for all } 1 \le l \le k-2. \tag{65}$$

Applying the general Leibniz product rule to the second term on the right-hand side in equation (59), noting that the first derivative of the second factor in this term vanishes, and using (64), the only non-vanishing term in the $l-1$th derivative comes from taking the $l-1$th derivative of the first factor. Analogous reasoning applies to the third term on the right-hand side in (59). Applying the general Leibniz rule to the fourth and fifth terms and using (14), the $l-1$th derivative of the sum of these terms converges to the $l$th derivative of $c$. In summary,

$$\begin{aligned}\hat{W}^{(l)} &:= \left.\frac{\mathrm{d}^l\hat{W}}{\mathrm{d}(\tau^S)^l}\right|_{\tau^S=F(0)} \\ &= \left.\frac{\mathrm{d}^l\rho(A)}{\mathrm{d}(\tau^S)^l}\right|_{\tau^S=F(0)} E[v] + (\Delta^R)^{(l)}\int_0^{\overline{v}} v\mathrm{d}F(v) + (\Delta^S)^{(l)}\int_{\underline{v}}^0 (-v)\mathrm{d}F(v) - c^{(l)}. \\ &\quad \text{for all } l = 1,\ldots,k.\end{aligned} \tag{66}$$

Using this together with (63), (64) and (65),

$$\hat{W}^{(l)} = 0 \quad \text{for all } 1 \le l \le k-2. \tag{67}$$

Using (60), (61), (62), and (14),

$$\begin{aligned}\left.\frac{\mathrm{d}^{k-1}\rho(A)}{\mathrm{d}(\tau^S)^{k-1}}\right|_{\tau^S=F(0)} &= \rho(A)_{(k-1)\tau^S} \overset{(42)}{=} k^S_{(k-1)\tau^S},\\ (\Delta^R)^{(k-1)} &= k^R_{(k-1)\tau^S} \overset{(40)}{=} -k^S_{(k-1)\tau^S},\\ (\Delta^S)^{(k-1)} &= k^S_{(k-1)\tau^S}.\end{aligned}$$

Thus, (66) implies

$$\hat{W}^{(k-1)} = 0. \tag{68}$$

Using the continuously differentiable inverse function $\tau^S(c)$ instead of $c(\tau^S)$, we can compute

$$W'(0) = \lim_{c\to 0} W'(c) = \lim_{c\to 0} \hat{W}'(\tau^S(c)) \cdot (\tau^S)'(c),$$

where we carefully distinguish $W$ (welfare as a function of $c$) from $\hat{W}$ (welfare as a function of $\tau^S$). Moving on,

$$\begin{aligned}W'(0) &= \lim_{c\to 0} \frac{\hat{W}'(\tau^S(c))}{c'(\tau^S(c))}\\ &= \lim_{\tau^S\to F(0)} \frac{\hat{W}'(\tau^S)}{c'(\tau^S)}.\end{aligned}$$

Due to (67), (68), and (14), we can apply L'Hospital's rule $k-1$ times, so that

$$W'(0) = \frac{\hat{W}^{(k)}}{c^{(k)}}, \tag{69}$$

where the indices in brackets denote the $k$th derivative at $\tau^S = F(0)$.

In order to compute $\hat{W}^{(k)}$, we have to compute the $l = k$th-order derivatives on the right-hand side of (66). Note that

$$\begin{aligned}\left.\frac{\mathrm{d}^{k}\rho(A)}{\mathrm{d}(\tau^S)^{k}}\right|_{\tau^S=F(0)} &= (\tau^R)^{(k)} \cdot \rho(A)_{(1)\tau^R} + \rho(A)_{(k)\tau^S}\\ &= x\mathbf{1}_{k>2}\frac{(n-t^*)k}{F(0)}\\ &\quad -x\mathbf{1}_{M_{t^*-2,\hat{k}}=1} \cdot \frac{t^*-1}{1-F(0)} + x\frac{\hat{k}(k-1)}{F(0)},\end{aligned} \tag{70}$$

where the first equation follows from the chain rule, the general Leibniz rule, and (14), and the second equation follows from (42), (43), and (73).

Similarly,

$$\begin{aligned}(\Delta^S)^{(k)} &= (\tau^R)^{(k)} \cdot k^S_{(1)\tau^R} + k^S_{(k)\tau^S}, \\ &= -x\mathbf{1}_{k=2}\frac{(n-t^*)k}{F(0)} \\ &\quad -x\mathbf{1}_{M_{t^*-2,\hat{k}}=1} \cdot \frac{t^*-1}{1-F(0)} + x\frac{k\hat{k}}{F(0)}, \end{aligned} \tag{71}$$

where the derivatives that occur on the right-hand side of (71) have been computed in (38), (39), and (73).

Similarly,

$$\begin{aligned}(\Delta^R)^{(k)} &= (\tau^R)^{(k)} \cdot k^R_{(1)\tau^R} + k^R_{(k)\tau^S}, \\ &= x \cdot \frac{(t^*-1)k}{1-F(0)} - x\mathbf{1}_{k>2} \cdot \frac{(n-t^*)k}{F(0)} \\ &\quad -x\mathbf{1}_{M_{t^*-2,\hat{k}}=0} \cdot \frac{t^*-1}{1-F(0)} - x\frac{\hat{k}(k-1)}{F(0)}, \end{aligned} \tag{72}$$

where the derivatives that occur on the right-hand side of (72) have been computed in (40), (47), and (73).

Plugging (70), (71), (72) into (66) at $l = k$, and that together with (73) into

(69), the variable $x$ cancels out and we find

$$\begin{aligned} W'(0) &= -\frac{f(0)}{k}\left(\mathbf{1}_{k>2}\frac{(n-t^*)k}{F(0)}\right. \\ &\qquad \left. -\mathbf{1}_{M_{t^*-2,\hat{k}}=1}\cdot\frac{t^*-1}{1-F(0)}+\frac{\hat{k}(k-1)}{F(0)}\right)E[vg(v)] \\ &\quad -\frac{f(0)}{k}\left(\frac{(t^*-1)k}{1-F(0)}-\mathbf{1}_{k>2}\cdot\frac{(n-t^*)k}{F(0)}\right. \\ &\qquad \left. -\mathbf{1}_{M_{t^*-2,\hat{k}}=0}\frac{t^*-1}{1-F(0)}-\frac{\hat{k}(k-1)}{F(0)}\right)\int_0^{\overline{v}} vg(v)\mathrm{d}F(v) \\ &\quad -\frac{f(0)}{k}\left(-\mathbf{1}_{k=2}\frac{(n-t^*)k}{F(0)}\right. \\ &\qquad \left. -\mathbf{1}_{M_{t^*-2,\hat{k}}=1}\cdot\frac{t^*-1}{1-F(0)}+\frac{k\hat{k}}{F(0)}\right)\int_{\underline{v}}^0 (-v)g(v)\mathrm{d}F(v) \\ &\quad -1. \end{aligned}$$

The terms with $\mathbf{1}_{k>2}$ and $\mathbf{1}_{k=2}$ can be summarized into a single term. Thus

$$\begin{aligned} W'(0) &= -\frac{f(0)}{k}\left(-\mathbf{1}_{M_{t^*-2,\hat{k}}=1}\cdot\frac{t^*-1}{1-F(0)}+\frac{\hat{k}(k-1)}{F(0)}\right)E[vg(v)] \\ &\quad -\frac{f(0)}{k}\left(\frac{(t^*-1)k}{1-F(0)}\right. \\ &\qquad \left. -\mathbf{1}_{M_{t^*-2,\hat{k}}=0}\frac{t^*-1}{1-F(0)}-\frac{\hat{k}(k-1)}{F(0)}\right)\int_0^{\overline{v}} vg(v)\mathrm{d}F(v) \\ &\quad -\frac{f(0)}{k}\left(-\frac{(n-t^*)k}{F(0)}\right. \\ &\qquad \left. -\mathbf{1}_{M_{t^*-2,\hat{k}}=1}\cdot\frac{t^*-1}{1-F(0)}+\frac{k\hat{k}}{F(0)}\right)\int_{\underline{v}}^0 (-v)g(v)\mathrm{d}F(v) \\ &\quad -1. \end{aligned}$$

Similarly, the terms with $\mathbf{1}_{M_{t^*-2,\hat{k}}=0}$ and with $\mathbf{1}_{M_{t^*-2,\hat{k}}=1}$ can be summarized into a single term. Thus,

$$\begin{aligned}
W'(0) &= -\frac{f(0)}{k}\frac{\hat{k}(k-1)}{F(0)}E[vg(v)] \\
&\quad -\frac{f(0)}{k}\left(\frac{(t^*-1)k}{1-F(0)}-\frac{t^*-1}{1-F(0)}-\frac{\hat{k}(k-1)}{F(0)}\right)\int_0^{\overline{v}} vg(v)\mathrm{d}F(v) \\
&\quad -\frac{f(0)}{k}\left(-\frac{(n-t^*)k}{F(0)}+\frac{k\hat{k}}{F(0)}\right)\int_{\underline{v}}^0 (-v)g(v)\mathrm{d}F(v) \\
&\quad -1. \\
&= -\frac{f(0)}{k}\left(\frac{(t^*-1)k}{1-F(0)}-\frac{t^*-1}{1-F(0)}\right)\int_0^{\overline{v}} vg(v)\mathrm{d}F(v) \\
&\quad -\frac{f(0)}{k}\left(-\frac{(n-t^*)k}{F(0)}+\frac{\hat{k}}{F(0)}\right)\int_{\underline{v}}^0 (-v)g(v)\mathrm{d}F(v) \\
&\quad -1 \\
&= f(0)\left(-\frac{t^*-1}{1-F(0)}+\frac{t^*-1}{k(1-F(0))}\right)\int_0^{\overline{v}} vg(v)\mathrm{d}F(v) \\
&\quad +f(0)\left(\frac{n-t^*}{F(0)}-\frac{n-t^*-k+1)}{kF(0)}\right)\int_{\underline{v}}^0 (-v)g(v)\mathrm{d}F(v) \\
&\quad -1 \\
&= f(0)\left(-\frac{t^*-1}{1-F(0)}+\frac{t^*-1}{k(1-F(0))}\right)\int_0^{\overline{v}} vg(v)\mathrm{d}F(v) \\
&\quad +f(0)\left(\frac{n-t^*+1}{F(0)}-\frac{n-t^*+1}{kF(0)}\right)\int_{\underline{v}}^0 (-v)g(v)\mathrm{d}F(v) \\
&\quad -1.
\end{aligned}$$

Thus, (17) holds. This completes the proof.

*Sketch of proof of Lemma 5.* We will use the shortcut

$$\hat{x} = \frac{(n-1)!}{r^*!(n-r^*-q)!}(1-F(0))^{r^*}F(0)^{n-q-r^*}.$$

Observe that

$$k^S = \left(\sum_{r+s\leq n-1,\ s\geq q} \dots\right) + \begin{pmatrix} & n-1 \\ r^* & q-1 \end{pmatrix} (\tau^R)^{r^*}(\tau^S)^{q-1}(1-\tau^R-\tau^S)^{n-q-r^*}.$$

Thus, using the general Leibniz product rule, and using the lower-case notation to denote higher-order partial derivatives evaluated at the point $(\tau^R, \tau^S) = (1 - F(0), 0)$,

$$k^S_{(l)\tau^S} = \begin{cases} 0 & \text{if } l \leq q-2, \\ \underbrace{\begin{pmatrix} n-1 \\ r^* \;\; q-1 \end{pmatrix}}_{=\frac{(n-1)!}{r^*!(n-r^*-q)!}} (q-1)!(1-F(0))^{r^*} F(0)^{n-q-r^*} & \text{if } l = q-1. \end{cases}$$

Thus, defining $\phi$ as in the proof of Lemma 3, we find $\phi_{(l)\tau^S} = (0,0)^T$ for all $l \leq q-2$, by arguments similar to the arguments in that proof. Thus, by arguments as in that proof, the first $l \leq q-2$ derivatives of the functions $c(\tau^S)$ and $\tau^R(\tau^S)$ vanish at $\tau^S = 0$. Moreover, again similar to that proof,

$$\begin{pmatrix} \left.\frac{\mathrm{d}^{q-1}c}{\mathrm{d}(\tau^S)^{q-1}}\right|_{\tau^S=0} \\ \left.\frac{\mathrm{d}^{q-1}\tau^R}{\mathrm{d}(\tau^S)^{q-1}}\right|_{\tau^S=0} \end{pmatrix} = (-\underline{v}) k^S_{(q-1)\tau^S} \begin{pmatrix} 1 \\ -\frac{f(0)}{\Delta^{t*}} \end{pmatrix} \tag{73}$$

Thus, $c(\tau^S)$ is strictly increasing in some right-interval of 0 and $\tau^R$ is strictly decreasing in some right-interval of 0. In summary, a result analogous to Lemma 3 holds. In other words, there exists indeed an equilibrium with small participation of the $S$-voters and large participation of the $R$-voters.

To verify the formula for the welfare effect, (22), we need

$$\begin{aligned} \rho(A) &= \sum_{r+s\leq n-1, s\geq q} (\dots) + \sum_{r+s\leq n-1, r\geq r^*, s\leq q-1} \begin{pmatrix} n-1 \\ r \; s \end{pmatrix} (\tau^R)^r (\tau^S)^s (1-\tau^R-\tau^S)^{n-1-r-s} \\ &= \sum (\dots) + \sum_{s=0}^{q-1} \sum_{r=r^*}^{n-1-s} \begin{pmatrix} n-1 \\ r \; s \end{pmatrix} (\tau^R)^r (\tau^S)^s (1-\tau^R-\tau^S)^{n-1-r-s}. \end{aligned}$$

Evaluating the $l$th partial derivative with respect to $\tau^S$ at the point $(1-F(0), 0)$, we find

$$\rho(A)_{(l)\tau^S} = \sum_{s=0}^{l} \sum_{r=r^*}^{n-1-l} \begin{pmatrix} n-1 \\ r \; s \end{pmatrix} (1-F(0))^r \begin{pmatrix} l \\ s \end{pmatrix} s! \frac{(n-1-r-s)!}{(n-1-r-l)!} (-1)^{l-s} F(0)^{n-1-r-l}.$$

(To see this, it is sufficient to sum up to $r = n-1-l$, observe that all terms vanish for an $r$ with $l > n-1-r = s + (n-1-r-s)$.)

After canceling some terms we obtain

$$\rho(A)_{(l)\tau^S} = \sum_{r=r^*}^{n-1-l} (1-F(0))^r F(0)^{n-1-r-l} \frac{(n-1)!}{r!(n-1-r-l)!} \underbrace{\sum_{s=0}^{l} \binom{l}{s} (-1)^{l-s}}_{=0}.$$
$$= 0.$$

Next,

$$\begin{aligned}
\rho(A)_{(1)\tau^R} &= \sum_{r=r^*}^{n-1} \binom{n-1}{r} \big(r(1-F(0))^{r-1}F(0)^{n-1-r} - (n-1-r)(1-F(0))^r F(0)^{n-2-r}\big) \\
&= \sum_{r=r^*}^{n-1} \frac{(n-1)!}{(r-1)!(n-2-(r-1))!}(1-F(0))^{r-1}F(0)^{n-2-(r-1)} \\
&\quad -\sum_{r=r^*}^{n-2} \frac{(n-1)!}{r!(n-2-r)!}(1-F(0))^r F(0)^{n-2-r} \\
&= \frac{(n-1)!}{(r^*-1)!(n-1-r^*)!}(1-F(0))^{r^*-1}F(0)^{n-1-r^*} \\
&= \Delta^{t*}\frac{n-q}{F(0)}.
\end{aligned}$$

Also,

$$k^R = \left(\sum_{r+s\leq n-1,\ s\geq q} \dots\right) + \sum_{s=0}^{q-1} \binom{n-1}{r^*-1\ \ s} (\tau^R)^{r^*}(\tau^S)^s(1-\tau^R-\tau^S)^{n-r^*-s}.$$

Thus, using the general Leibniz product rule, for all $l \leq q-1$,

$$\begin{aligned}
k^R_{(l)\tau^S} &= \sum_{s=0}^{l} \binom{n-1}{r^*\ \ s} (1-F(0))^{r^*-1} \binom{l}{s} s!\frac{(n-r^*-s)!}{(n-r^*-l)!}(-1)^{l-s}F(0)^{n-r^*-l} \\
&= \frac{(n-1)!}{(r^*-l)!(n-r^*-l)!}(1-F(0))^{r^*-1}F(0)^{n-r^*-l} \underbrace{\sum_{s=0}^{l}(-1)^{l-s}}_{=0} \\
&= 0.
\end{aligned}$$

Also,

$$\begin{aligned} k^R_{(1)\tau^R} &= \begin{pmatrix} n-1 \\ r^*-1\,0 \end{pmatrix} \left((r^*-1)(1-F(0))^{r^*-2}F(0)^{n-r^*} - (1-F(0))^{r^*-1}(n-r^*)F(0)^{n-r^*-1}\right) \\ &= \Delta^{t*}\left(\frac{r^*-1}{1-F(0)} - \frac{n-r^*}{F(0)}\right). \end{aligned}$$

We are now prepared to compute the welfare effect $W'(0)$.

From the computations above it follows that formulas (63) (64), and (65) still hold, with derivatives evaluated at $\tau^S = 0$ instead of at $1-F(0)$. We are using the upper index $(l)$ to denote the $l$th derivative evaluated at $\tau^S = 0$.

Formula (66) is replaced by

$$\hat{W}^{(l)} := \left.\frac{\mathrm{d}^l \hat{W}^M}{\mathrm{d}(\tau^S)^l}\right|_{\tau^S=0} = \left.\frac{\mathrm{d}^l \rho(A)}{\mathrm{d}(\tau^S)^l}\right|_{\tau^S=0} E[vg(v)] + (\Delta^R)^{(l)} \int_0^{\overline{v}} vg(v)\mathrm{d}F(v) - \int_0^{\overline{v}} g(v)\mathrm{d}F(v)c^{(l)}.$$
$$\text{for all } l = 1, \ldots, q.$$

Thus, (67) still holds (again evaluating the derivative at $\tau^S = 0$).

Moreover, plugging in what we know,

$$\begin{aligned} \hat{W}^{(q-1)} &= \Delta^{R*}\frac{n-r^*}{F(0)}(-\underline{v})k^S_{(q-1)\tau^S}\frac{-f(0)}{\Delta^{t*}}E[vg(v)] \\ &\quad +\Delta^{t*}\left(\frac{r^*-1}{1-F(0)} - \frac{n-r^*}{F(0)}\right)(-\underline{v})k^S_{(q-1)\tau^S}\frac{-f(0)}{\Delta^{t*}}\int_0^{\overline{v}} vg(v)\mathrm{d}F(v) \\ &\quad -\int_0^{\overline{v}} g(v)\mathrm{d}F(v)c^{(q-1)}. \end{aligned}$$

Here, $\Delta^{t*}$ cancels out. Eventually,

$$\begin{aligned} W'(0) &= \frac{\hat{W}^{(q-1)}}{c^{(q-1)}} \\ &= -f(0)\frac{n-r^*}{F(0)}E[vg(v)] \\ &\quad -f(0)\left(\frac{r^*-1}{1-F(0)} - \frac{n-r^*}{F(0)}\right)\int_0^{\overline{v}} vg(v)\mathrm{d}F(v) - \int_0^{\overline{v}} g(v)\mathrm{d}F(v), \\ &= f(0)\left((n-r^*)E_- - (r^*-1)E_+\right) - \int_0^{\overline{v}} g(v)\mathrm{d}F(v). \end{aligned}$$

showing (22). This completes the proof.

*Sketch of proof of Lemma 6.* Define the function

$$\phi(c, \tau^R) \;\; = \;\; F^{-1}(1-\tau^R)k^R(\tau^R) - c,$$

where

$$k^R(\tau^R) \;\; = \;\; \left( \begin{array}{c} n-1 \\ r^*-1 \end{array} \right) (\tau^R)^{r^*-1}(1-\tau^R)^{n-r^*}.$$

Because

$$\frac{\partial \phi}{\partial \tau^R}|_{c=0,\tau^R=1-F(0)} \;\; = \;\; -\frac{1}{f(0)}k^R(1-F(0)) = -\frac{\Delta^{r*}}{f(0)} < 0,$$

we can apply the IFT and find a function $\tau^R(c)$ for $c$ close to 0 satisfying $\phi(c, \tau^R(c)) = 0$ and

$$(\tau^R)'(c) \;\; = \;\; \frac{1}{\frac{\partial \phi}{\partial \tau^R}},$$

implying

$$(\tau^R)'(0) \;\; = \;\; -\frac{f(0)}{\Delta^{r*}} < 0.$$

It remains to prove (22). Observe that

$$W(c) \;\; = \;\; \rho(A)(c)E[vg(v)] + k^R(\tau^R(c)) \int_{F^{-1}(1-\tau^R(c))}^{\overline{v}} vg(v)\mathrm{d}F(v) - \int_{F^{-1}(1-\tau^R(c))}^{\overline{v}} g(v)\mathrm{d}F(v) \cdot c,$$

where

$$\rho(A)(c) \;\; = \;\; \sum_{r=r^*}^{n-1} \left( \begin{array}{c} n-1 \\ r \end{array} \right) (\tau^R(c))^r(1-\tau^R(c))^{n-1-r}.$$

Therefore, using the shortcut

$$\delta_r \;\; = \;\; \left( \begin{array}{c} n-1 \\ r \end{array} \right) \left( r(1-F(0))^{r-1}F(0)^{n-1-r} - (1-F(0))^r(n-1-r)F(0)^{n-2-r} \right),$$

we find

$$W'(0) \quad = \quad \sum_{r=r^*}^{n-1} \delta_r \cdot (\tau^R)'(0)E[vg(v)] + \delta_{r^*-1} \cdot (\tau^R)'(0) \int_0^{\overline{v}} vg(v)\mathrm{d}F(v) - \int_0^{\overline{v}} g(v)\mathrm{d}F(v).$$

Note that

$$\delta_{r^*-1} \quad = \quad \Delta^{r*}\left(\frac{r^*-1}{1-F(0)} - \frac{n-r^*}{F(0)}\right)$$

and

$$\begin{aligned} &\sum_{r=r^*}^{n-1} \begin{pmatrix} n-1 \\ r \end{pmatrix} r(1-F(0))^{r-1}F(0)^{n-1-r} \\ = &\sum_{\hat{r}=r^*-1}^{n-2} \begin{pmatrix} n-1 \\ \hat{r} \end{pmatrix} (n-1-\hat{r})(1-F(0))^{\hat{r}}F(0)^{n-2-\hat{r}}, \end{aligned}$$

implying

$$\sum_{r=r^*}^{n-1} \delta_r = \begin{pmatrix} n-1 \\ r^*-1 \end{pmatrix} (n-r^*)(1-F(0))^{r^*-1}F(0)^{n-r^*-r} = \Delta^{r*}\frac{n-r^*}{F(0)}.$$

Thus,

$$\begin{aligned} W'(0) \quad &= \quad -\frac{f(0)}{\Delta^{r*}}\left(\Delta^{r*}\frac{n-r^*}{F(0)}E[vg(v)] + \Delta^{r*}\left(\frac{r^*-1}{1-F(0)} - \frac{n-r^*}{F(0)}\right)\int_0^{\overline{v}} vg(v)\mathrm{d}F(v)\right) \\ &\quad - \int_0^{\overline{v}} g(v)\mathrm{d}F(v) \\ &= \quad -f(0)\frac{n-r^*}{F(0)}E[vg(v)] - f(0)\left(\frac{r^*-1}{1-F(0)} - \frac{n-r^*}{F(0)}\right)\int_0^{\overline{v}} vg(v)\mathrm{d}F(v) \\ &\quad - \int_0^{\overline{v}} g(v)\mathrm{d}F(v), \end{aligned}$$

showing (22).

# References


Aguiar-Conraria, L., and Magalhães, P. C. (2010). How quorum rules distort referendum outcomes: Evidence from a pivotal voter model. European Journal of Political Economy, 26(4), 541-557.

Azrieli, Y., and S. Kim (2014), "Pareto efficiency and weighted majority rules," *International Economic Review* **55**, 1067–1088.

Barberà, S., and M.O. Jackson (2006), "On the Weights of Nations: Assigning Voting Weights in a Heterogeneous Union," *Journal of Political Economy* **114**, 317–339.

Bognar, K., T. Börgers, and M. Meyer-ter-Vehn (2015), "An optimal Voting System when Voting is costly," *Journal of Economic Theory* **159**, 1056–1073.

Börgers, T. (2000), "Costly Voting," *Working Paper*.

Börgers, T. (2004), "Costly Voting," *American Economic Review* **94**, 57–66.

Campbell, C. M. (1999), "Large electorates and decisive minorities," *Journal of Political Economy* **107**, 1199–1217.

Chakravarty, S., Kaplan, T. R., and Myles, G. D. (2010). The benefits of costly voting. Available at SSRN 1570094.

Charlétyy, P., M.-C. Fagart, and S. Souam (2015), Quorum Rules and Shareholder Power, mimeo, Paris Descartes.

Côrte-Real, P. P., and Pereira, P. T. (2004). The voter who wasn't there: referenda, representation and abstention. Social Choice and Welfare, 22(2), 349-369.

Das, S. (2014). Town-Hall Meetings in Electoral Democracy: Theory and Evidence from Indian Village Councils, mimeo, Yale University.

Feddersen, T. J., and Pesendorfer, W. (1996). The swing voter's curse. The American Economic Review, 408-424.

Feddersen, T., and A. Sandroni (2006), "A theory of participation in elections," *The American Economic Review* **96**, 1271–1282.

Gershkov, A., B. Moldovanu, and X. Shi (2013), "Optimal mechanism design without money," working paper, University of Bonn.

Ghosal, S., and Lockwood, B. (2009). Costly voting when both information and preferences differ: is turnout too high or too low?, Social Choice and Welfare, 33, 25–50.

Goeree, J. K., and J. Großer (2007), "Welfare reducing polls," *Economic Theory* **31**, 51–68.

Herrera and Matozzi, 2008, Quorum and Turnout in Referendum, http://tintin.hec.ca/pages/helios.herrera

Kartal, M. (2014), "A comparative welfare analysis of electoral systems with endogenous turnout." *The Economic Journal.*

Kos, N. (2012), "Communication and Efficiency in Auctions," *Games and Economic Behavior* **75**, 233–249.

Krasa, S., and M. Polborn (2009), "Is Mandatory Voting Better than Voluntary Voting?," *Games and Economic Behavior* **66**, 275–291.

Krishna, V., and Morgan, J. (2012). Majority rule and utilitarian welfare. Available at SSRN 2083248.

Laruelle, A., and F. Valenciano (2011), "Majorities with a quorum," *Journal of Theoretical Politics* **23**, 241–259.

Ledyard, J.O. (1984), "The pure theory of large two-candidate elections," *Public Choice* **44**, 7–41.

Myatt, D.P., 2014, "A Rational Choice Theory of Voter Turnout," *Working paper*, London Business School.

Palfrey, T. R., and Rosenthal, H. (1983). A strategic calculus of voting. Public Choice, 41(1), 7-53.

Palfrey, T. R., and H. Rosenthal (1985), "Voter participation and strategic uncertainty," *American Political Science Review* **79**, 62–78.

Rae, D. W. (1969). Decision-rules and individual values in constitutional choice. American Political Science Review, 63(01), 40-56.

Schmitz, P. W., and T. Tröger (2012), "The (sub-) optimality of the majority rule," *Games and Economic Behavior* **74**, 651–665.

Taylor, CR and H Yildirim (2010b) A unified analysis of rational voting with private values and group-specific costs, Games and Economic Behavior 70 (2), 457-471.